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CLASSICAL PROBLEMS
OF LINEAR ACOUSTICS AND WAVE THEORY

Acoustic Scattering on Spheroidal Shapes Near Boundaries[1]

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Abstract—A new expression for the Lamé product of prolate spheroidal wave functions is presented in terms of a distribution of multipoles along the axis of the spheroid between its foci (generalizing a corresponding theorem for spheroidal harmonics). Such an "ultimate" singularity system can be effectively used for solving various linear boundary-value problems governed by the Helmholtz equation involving prolate spheroidal bodies near planar or other boundaries. The general methodology is formally demonstrated for the axisymmetric acoustic scattering problem of a rigid (hard) spheroid placed near a hard/soft wall or inside a cylindrical duct under an axial incidence of a plane acoustic wave.

Keywords: Linear acoustics and Helmholtz equation, spheroidal wave functions, multipole expansions and ultimate singularity system, Green's function and integral representation, planar boundaries and cylindrical duct

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1. INTRODUCTION

Recently (see, for example, [1, 2]), we have analytically considered several practical aspects of free-surface hydrodynamics (e.g., wave resistance, wave diffraction etc.) of prolate spheroidal-like rigid vessels moving in some bounded flow domains (i.e., shallow water, near a vertical bank or along a straight channel). The field equation in these cases (incompressible fluid and irrotational flow), is the Laplace equation, and the corresponding linear boundary-value problem involving a spheroidal rigid hull is solved using elementary spheroidal harmonics expressed in terms of the Legendre' associate polynomials [3, 4]. However, if some nearby planar boundaries (i.e., vertical wall, flat bottom or free surface) are present, the solution of the corresponding hydrodynamic problem becomes more intricate and in general cannot be determined analytically (compared to the unbounded case). The confined problem can still be numerically resolved by solving a set of coupled Fredholm integral equations of the second- kind using a specific (e.g., finite element or finite difference) numerical scheme. Nevertheless, it has been demonstrated in [1, 2] that by using a special theorem [5], which expresses any external spheroidal harmonic in terms of its "ultimate" image singularity system (consisting of interior multipoles distributed along the body axis), one can still get a semi-analytic solution even for the case of a spheroid moving in the proximity of other planar bounding surfaces. A similar approach can be also used for other physical problems governed by the Laplace equation (e.g., inviscid hydrodynamics, heat conduction, electrostatic, electrokinetics etc.), entailing tri-axial ellipsoidal objects [6], thus enabling us to consider some new practical problems of mathematical-physics.

In this note, we are mainly interested in the wide class of 3D wave scattering problems involving prolate spheroidal (including penetrable and elastic) shapes, where the field equation is governed by the Helmholtz rather than by the Laplace's equation [7], such as acoustic and electromagnetic wave scattering by prolate and tri-axial ellipsoidal scatters [8–12], electrostatic [13], electrophoresis [14], creeping flows [15], antennas [16], nuclear physics [17], optics [18], etc. The traditional way for solving the Helmholtz equation, which is subject to some linear boundary conditions (i.e., Dirichlet, Neumann or Robin) applied on spheroidal shapes, generally employ the so-called "spheroidal wave functions" (SWF) which have been extensively explored in the literature [19–24]. Indeed, spheroidal (prolate or oblate) wave functions find many applications in science and technology, mainly for computing wave diffraction from objects of various sizes (spanning from nuclear physics to cosmology). In this context, we mention in particular the case of sound backscattering from spheroidal shapes of different surface properties (coating) due to some external acoustic excitations and disturbances.

The salient acoustic problem of wave scattering and diffraction from a spheroidal-like vessel lying in an infinite domain which is exposed to an incident plane (monochromatic) sound wave can be naturally

[1] The article is published in the original.

resolved by using expansions in terms of SWF. The first to analytically tackle this problem by using SWF based techniques, were probably Spence and Granger [25], but their solution was restricted to obtaining only the far-field acoustic scattering (cross-section) patterns. The same problem has continued to attract attention over the years (even to date) and there is a vast body of literature on SWF (see, for example, [26–44]). Nevertheless, it is important to note that most of these studies still deal with acoustic scattering from a single spheroidal shape, namely without considering mutual interaction effects due to nearby boundaries, free surfaces or other interacting obstacles on the resulting wave scattering. In order to analytically resolve such problems, it is advantageous to first obtain (in a similar manner to [1, 2]), a corresponding expression for the "ultimate" singularity system of any external SWF eigensolution. Here we derive an explicit expression which provides a generalization of the Laplacian so-called "spheroid theorem" [5] for the more general Helmholtz equation. A similar expression is also obtained in this note as a limiting case for spherical shapes and is compared against the corresponding multipole expansion given in Hobson [3].

The structure of the paper is as follows: In Sect. 2 we briefly discuss the various angular and radial SWF using the notation and terminology of Flammer [20]. The known expressions for the "ultimate" (internal) system of singularities (multipoles) for both spherical and spheroidal (Laplacian) harmonics, are re-derived and explicitly given in Sect. 3, together with their new extensions for physical cases governed by the Helmholtz field equation. In particular, we refer to Eqs. (10) and (27), which provide explicit expressions for any external separable eigensolutions, in terms of a distribution of multipoles lying at the origin or along the axis between the two foci. The same integral expression is also shown to be amenable for obtaining a useful asymptotic form for the far-field sound scattering (cross-section). The above methodology is finally demonstrated in Sect. 4 for the special axisymmetric case of sound scattering from a rigid spheroid placed near a planar (hard or soft) wall and subject to an incident axial plane acoustic wave. Yet another practical acoustic example, involving wave scattering of a hard spheroid placed on the axis of a cylindrical duct, is presented in Sect. 5. In both cases we choose to dwell here only on analytic rather than on numerical aspects and derive the corresponding set of linear equations for the amplitudes of the scattered wave field. A closed form leading-order asymptotic solution of these equations is presented under the "large-spacing" approximation. Detailed numerical wave-scattering simulations for some factual cases based on the above methodology, including non-symmetric (oblique incidence) scenarios, will be presented elsewhere.

2. SPHEROIDAL WAVE FUNCTIONS

The following Helmholtz or "wave" equation arises in many branches of mathematical physics (e.g. [7]), such as acoustics, electromagnetic diffractions, low-Reynolds number flows and induced-charge electro-osmosis:

$$\left(\nabla^2 + k^2\right)\phi = 0, \tag{1}$$

where the parameter k can be real or complex. In this work we are interested in obtaining separable solutions of (1) for the real function ϕ in terms of the triply orthogonal prolate spheroidal coordinate system (μ, ζ, φ) which are related to the Cartesian (x, y, z) ones by

$$x = d\mu\zeta,\ y + iz = d\left(1-\mu^2\right)^{1/2}\left(\zeta^2 - 1\right)^{1/2} e^{i\varphi}. \tag{2}$$

Here d represents half the distance between the two spheroid foci: $1 \ge \mu \ge -1$, $\infty \ge \zeta \ge 1$ and $2\pi \ge \varphi \ge 0$. Denoting the major and semi axes of the spheroid by a and b respectively, then $d = \left(a^2 - b^2\right)^{1/2}$ and $\zeta_0 = \left(1 - b^2/a^2\right)^{-1/2} > 1$ represents the surface of the spheroid.

Using Flammer's [20] notations, it is possible to express general separable solutions of (1) by using the so-called *"spheroidal wave functions"*, which depend on the spheroidal coordinates and the dimensionless parameter: $c = kd$. For example, an "interior" eigensolution of (1), which is regular at the origin and diverges far from it, can be written as $S_{mn}(c,\mu) R^{(1)}_{mn}(c,\zeta) e^{im\varphi}$. In a similar manner, the corresponding "exterior" eigensolution of (1), which is singular at the origin and vanishes at infinity, is given by $S_{mn}(c,\mu) R^{(3)}_{mn}(c,\zeta) e^{im\varphi}$. Here m, n denote any two independent positive integers, $S_{mn}(c,\mu)$ represents the "angular" SWF of the first- kind and $R^{(1,3)}_{mn}(c,\zeta)$ are the corresponding "radial" SWF of the first and second kind respectively. These functions have been discussed and analyzed in great length in several texts (see, for example, [7, 19–24]) and for reasons of brevity details are not repeated here. Nevertheless, it is important to mention in the present context that the angular functions $S_{mn}(c,\mu)$ are orthogonal over the interval $|\mu| < 1$ and can be expressed as an infinite series involving the common associate Legendre polynomials $P_n^m(\mu)$ and some prescribed coefficients (denoted in [16] as $\hat{d}_r^{mn}(c)$ where $r = 0, 1, 2, \ldots$). In a similar way, the two radial functions $R^{(1)}_{mn}(c,\zeta)$ and $R^{(3)}_{mn}(c,\zeta)$ can also be written as products of the same coefficients with the spherical Bessel or Hankel functions respectively. There exist several computational software for evaluating both the angular and radial

SWF for any order (m, n) and dimensionless parameter c [23, 40, 42].

The fundamental (Green's function) solution of the Helmholtz equation (1) can be expanded in terms of SWF (see [20], Eq. (5.2.9)) as

$$\frac{e^{ikR_{pq}}}{kR_{pq}} = 2i\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{\varepsilon_m}{N_{mn}}S_{mn}(c,\mu_p)S_{mn}(c,\mu_q) \times R_{mn}^{(1)}(c,\zeta_p)R_{mn}^{(3)}(c,\zeta_q)\cos m(\varphi_q-\varphi_p) \quad (3)$$

where $\zeta_q > \zeta_p$, $\varepsilon_0 = 1$ and $\varepsilon_m = 2$ for $m \neq 0$. Here R_{pq} represents the radial distance between an "interior" point $P(\mu_p, \zeta_p, \varphi_p)$ lying within the spheroid and some "external" point $Q(\mu_q, \zeta_q, \varphi_q)$. The coefficient N_{mn} in (3) is associated with the orthogonality properties of the angular wave functions and is defined by (see [20], Eqs. (3.1.32), (3.1.33))

$$\int_{-1}^{1} S_{mn}(c,\mu)S_{mn'}(c,\mu)d\mu = \delta_{nn'}N_{mn}, \quad (4)$$

and can be explicitly expressed (Eq. (3.1.33) of [20]) in terms of the coefficients $\hat{d}_r^{mn}(c)$. As $k \to 0$, the Helmholtz equation (1) reduces to the Laplace's equation, possessing the fundamental solution $1/R_{pq}$, which can be accordingly expressed in terms of the associated Legendre' functions $P_n^m(\mu)$ and $Q_n^m(\zeta)$ (see [3], p. 416) as

$$\frac{d}{R_{pq}} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(-1)^m\varepsilon_m(2n+1)\left[\frac{(n-m)!}{(n+m)!}\right]^2 \times P_n^m(\mu_p)P_n^m(\mu_q)P_n^m(\zeta_p)Q_n^m(\zeta_q)\cos m(\varphi_q-\varphi_p). \quad (5)$$

In comparing (3) with (5) for $k = c = 0$, one notes that $S_{mn}(0,\mu) \to P_n^m(\mu)$ and that $R_{mn}^{(1)}(0,\zeta)$ and $R_{mn}^{(3)}(0,\zeta)$ are proportional to $P_n^m(\zeta)$ and $Q_n^m(\zeta)$ respectively.

An equivalent representation in terms of spherical coordinates (R,μ,φ) can be also obtained directly from (2) by simply letting $d \to 0$ and $\zeta \to \infty$, such that $R = d\zeta$ denotes the radial (finite) coordinate. The corresponding expression for the Green's function is then given by (e.g., [7], Eq. (11.3.44))

$$\frac{e^{ikR_{pq}}}{kR_{pq}} = 2i\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varepsilon_m(2n+1)\frac{(n-m)!}{(n+m)!}P_n^m(\mu_p) \times P_n^m(\mu_q)j_n(kR_p)h_n^{(1)}(kR_q)\cos m(\varphi_q-\varphi_p), \quad (6)$$

where $j_n(z)$ and $h_n^{(1)}(z)$ represent the common spherical Bessel functions of the first and third (Hankel) kind respectively (see [21], p. 437).

3. EXPRESSIONS IN TERMS OF THE "ULTIMATE" SINGULARITY SYSTEM

3.1. Spherical Coordinates

A useful expression for any spherical external eigensolution $R^{-(n+1)}P_n^m(\mu)e^{im\varphi}$, satisfying the Laplace equation ($k = 0$) and expressed in terms of series of multipoles lying at the origin, can be obtained by a proper Cartesian differentiation of the fundamental Green's function $1/R$ (see [3], p. 134) as

$$\frac{P_n^m(\mu)e^{im\varphi}}{R^{n+1}} = \frac{(-1)^n}{(n-m)!}\frac{\partial^{n-m}}{\partial x^{n-m}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^m\frac{1}{R}. \quad (7)$$

The above relation suggests that any exterior spherical potential function, which decays to zero at infinity, can be represented by a series of multipoles $\dfrac{\partial^{k+m+n}}{\partial x^k\partial y^m\partial z^n}$ of order $k+m+n$ (k, m, n are arbitrary positive integers) located at the origin. This also means that the sphere's center (origin) constitutes the "ultimate" (or minimal) geometrical domain (a single point in this case), onto which all internal singularities of the external field can be condensed by analytic continuation.

It is interesting to note that an analogous expression to (7) for the Helmholtz equation does not exist even for the case of a perfectly symmetric sphere. Thus, by virtue of (7), let us postulate the following relation in terms of an "ultimate" multipole system for the Helmholtz wave equation (1):

$$k^{n+1}\tilde{A}_n^mP_n^m(\mu)h_n^{(1)}(kR)e^{im\varphi} = \frac{\partial^{n-m}}{\partial x^{n-m}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^m\left(\frac{e^{ikR}}{R}\right), \quad (8)$$

where $h_n^{(1)}(z)$ represents the spherical Bessel (Hankel) function (see [21], Ch. 10) and the coefficients $\tilde{A}_n^m$ are to be determined. First, we recall that both sides of (8) satisfy Eq. (1) when expressed in terms of spherical coordinates. Next, by using successive partial differentiation and the definition of the Hankel function, one can show that

$$\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^m\left(\frac{e^{ikR}}{R}\right) = (y+iz)^m\left(\frac{1}{R}\frac{d}{dR}\right)^m\left(\frac{e^{ikR}}{R}\right) = i(-1)^m k^{m+1}(y+iz)^m\frac{h_m^{(1)}(kR)}{R^m}, \quad (9)$$

which also implies that

$$\frac{\partial^{n-m}}{\partial x^{n-m}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^m\left(\frac{e^{ikR}}{R}\right) = i(-1)^m k^{m+1}R^m\left(1-\mu^2\right)^{m/2}e^{im\varphi}\frac{\partial^{n-m}}{\partial x^{n-m}}\left(\frac{h_m^{(1)}(kR)}{R^m}\right). \quad (10)$$

Selecting $\mu \to 1$ in (10) and denoting $\tilde{P}_n^m(\mu) = \left(1-\mu^2\right)^{-m/2} P_n^m(\mu)$, implies that along the axis of symmetry $R = x$, and thus (8) reduces to

$$\begin{aligned} &k^{n+1}\tilde{A}_n^m \tilde{P}_n^m(1) h_n^{(1)}(kx) \\ &= i(-1)^m k^{m+1} x^m \frac{d^{n-m}}{dx^{n-m}}\left(\frac{h_m^{(1)}(kx)}{x^m}\right). \end{aligned} \tag{11}$$

Letting next $x \to \infty$ ($k \neq 0$) in (11) and employing the following asymptotic expression for the Hankel function [21]:

$$\lim_{kx\to\infty} h_m^{(1)}(kx) \to (-i)^{m+1}\frac{e^{ikx}}{kx}, \tag{12}$$

one finally finds the unknown coefficient in (8) for $k \neq 0$:

$$\tilde{A}_n^m = \frac{i(-1)^n}{\tilde{P}_n^m(1)} = \frac{i(-1)^n 2^m m!(n-m)!}{(n+m)!}. \tag{13}$$

On the other hand, if we consider the limit $k \to 0$ in (11), then one gets

$$\lim_{kx\to 0} h_m^{(1)}(kx) \to -\frac{i(2m)!}{2^m m!}\frac{1}{(kx)^{m+1}} \tag{14}$$

and following (11) we obtain (for $k = 0$)

$$\tilde{A}_n^m = \frac{i2^n n!}{(2n)!}. \tag{15}$$

It is worth mentioning that (8) reduces to (7) for $k = 0$ by virtue of (14) and (15). Thus, Eqs. (8) and (13) provide the sought ultimate singularity expression for any external eigensolution of the Helmholtz wave equation (1) for $k \neq 0$, expressed in terms of a series of multipoles placed at the center of the sphere.

3.2. Prolate Spheroids

Our next task is to extend the above formulation for spheroidal coordinates and derive an analogous expression to (8) in terms of the appropriate ultimate singularity system for a prolate spheroidal shape. Towards this goal, let us define $R = R_{pq}$ as the distance between an arbitrary field point $Q(x,y,z)$ and an internal point $P(d\xi,0,0)$ where $|\xi| < 1$, which is located on the major axis of the spheroid between its foci, namely $R^2 = (x-d\xi)^2 + y^2 + z^2$. Furthermore, using (2) and (11), one gets

$$\begin{aligned} &\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)^m \left(\frac{e^{ikR}}{R}\right) \\ &= d^m\left(1-\mu^2\right)^{m/2}\left(\zeta^2-1\right)^{m/2} e^{im\varphi}\left(\frac{1}{R}\frac{d}{dR}\right)^m\left(\frac{e^{ikR}}{R}\right). \end{aligned} \tag{16}$$

Since the left hand side of (16) is a solution of the wave equation (1) which decays at infinity, it can be expanded in terms of SWF by employing symmetry arguments and making use of Eqs. (3), (5) and (6), as

$$\begin{aligned} &d^{m+1}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)^m\left(\frac{e^{ikR}}{R}\right) \\ &= \sum_{n=m}^{\infty}\frac{\tilde{B}_n^m}{N_{mn}}\tilde{S}_{mn}(c,\xi) S_{mn}(c,\mu)\tilde{R}_{mn}^{(3)}(c,\zeta) e^{im\varphi}, \end{aligned} \tag{17}$$

where, according to the present definition

$$\tilde{S}_{mn}(c,\mu) = \left(1-\mu^2\right)^{-m/2} S_{mn}(c,\mu),$$

$$\tilde{R}_{mn}^{(1,3)}(c,\zeta) = \left(\zeta^2-1\right)^{-m/2} R_{mn}^{(1,3)}(c,\zeta)$$

and the coefficients $\tilde{B}_n^m$ are to be determined. Applying next the orthogonality relation (4) for the angular SWF to (16) and (17), leads to

$$\begin{aligned} &i(-1)^m c^{2m+1}\int_{-1}^{1}\frac{h_m^{(1)}(kR)}{(kR)^m}\left(1-\xi^2\right)^{m/2} S_{mn}(\mathrm{c},\xi)\, d\xi \\ &= \tilde{B}_n^m \tilde{S}_{mn}(c,\mu)\tilde{R}_{mn}^{(3)}(c,\zeta), \end{aligned} \tag{18}$$

which results from the following definition of the spherical Hankel function [21]:

$$\left(\frac{1}{R}\frac{d}{dR}\right)^m\left(\frac{e^{ikR}}{R}\right) = i(-1)^m k^{2m+1}\frac{h_m^{(1)}(kR)}{(kR)^m}. \tag{19}$$

In order to determine the unknown coefficients $\tilde{B}_n^m$ in (18), we let $\mu \to 1$ so that following (2) $x = d\zeta$, which implies that $R = d(\zeta - \xi)$. If we further assume that $\zeta \to \infty$ and recall that $|\xi| \le 1$ and $c = kd$, one gets

$$\lim_{\zeta\to\infty}\frac{h_m^{(1)}(kR)}{(kR)^m} \to \frac{(-i)^{m+1}}{c^{m+1}}\frac{e^{ic(\zeta-\xi)}}{(\zeta-\xi)^{m+1}}. \tag{20}$$

Substituting (20) in (18) leads to

$$\begin{aligned} &\tilde{B}_n^m \tilde{S}_{mn}(c,1)\lim_{\zeta\to\infty}\left[\zeta^{m+1} e^{-ic\zeta}\tilde{R}_{mn}^{(3)}(c,\zeta)\right] \\ &= (ic)^m\int_{-1}^{1} e^{-ic\xi}\left(1-\xi^2\right)^{m/2} S_{mn}(c,\xi)\, d\xi. \end{aligned} \tag{21}$$

The integral on the right hand side of (21) can be evaluated by manipulation of Eq. (5.3.12) of [20], which renders

$$\begin{aligned} &c^m\int_{-1}^{1} e^{-ic\xi}\left(1-\xi^2\right)^{m/2} S_{mn}(c,\xi)\, d\xi \\ &= 2^{m+1} m!\, i^{n-m}\tilde{S}_{mn}(c,1)\tilde{R}_{mn}^{(1)}(c,1), \end{aligned} \tag{22}$$

where $\tilde{R}^{(1)}_{mn}(c,1) = \lim_{\zeta\to 1^+}\left(\zeta^2-1\right)^{-m/2} R^{(1)}_{mn}(c,\zeta)$. Using next the expression for the radial SWF $R^{(3)}_{mn}(c,\zeta)$ given in Eq. (4.1.17) of [20], one can show that

$$\lim_{\zeta\to\infty}\left[\zeta^{m+1}e^{-ic\zeta}\tilde{R}^{(3)}_{mn}(c,\zeta)\right] \to -\frac{i^{n+1}}{c}. \tag{23}$$

The unknown coefficient in (17) can then be explicitly obtained by substituting (22) and (23) in (21), yielding

$$\tilde{B}^m_n(c) = (ic)2^{m+1}m!\,\tilde{R}^{(1)}_{mn}(c,1). \tag{24}$$

Eq. (24) can be easily verified for the special case of $m = 0$ directly from Eqs. (3) and (17) by noting that $\tilde{B}^0_n(c) = 2ic\tilde{R}^{(1)}_{0n}(c,1)$. It is finally noted that (24) enables us to express any external eigensolution of Eq. (1) in the following integral form:

$$\begin{aligned}&\tilde{\mathrm{B}}^m_n(c)S_{mn}(c,\mu)R^{(3)}_{mn}(c,\zeta)e^{im\varphi}\\ &= d^{m+1}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^m\int_{-1}^{1}\frac{e^{ikR}}{R}\left(1-\xi^2\right)^{m/2}S_{mn}(c,\xi)\,d\xi,\end{aligned} \tag{25}$$

where the coefficients $\tilde{B}^m_n(c)$ are given by (24) and it is reminded that R denotes here the distance between any field point $Q(x,y,z)$ and an interior point $P(d\,\xi,0,0)$.

Eq. (25) is the sought expression for the exterior prolate spheroid potential of the Helmholtz equation, expressed in terms of an ultimate singularity system consisting of a distribution of multipoles of the fundamental Green's function e^{ikR}/R over the major axis of the spheroid. The coefficient $\tilde{R}^{(1)}_{mn}(c,1)$ in (24) is given explicitly in Eq. (4.6.11) of [20] for the case where the integer $n-m$ is either even or odd. It is not difficult to show that when the spheroid degenerates unto a sphere (i.e., $d\to 0$), Eq. (25) reduces to (8). This can be done by noting that as $c\to 0$ and $\zeta\to\infty$, one gets $R = d\zeta$ and $S_{mn}(c,\mu)\to P^m_n(\mu)$, $R^{(3)}_{mn}(c,\zeta)\to h^{(1)}_n(kR)$. The equivalence between (25) and (8) is then obtained by making use of (13) and (24) by recalling that $\tilde{R}^{(1)}_{mn}(c,1)\to \dfrac{c^n}{1\cdot 3\cdots(2n+1)}$ for $c\to 0$.

Consider next a "far-field" point $Q(x,y,z)$, such that its radial distance $\rho = \left(x^2+y^2+z^2\right)^{1/2}$, where $x = \rho\cos\theta$ and $y+iz = \rho\sin\theta e^{i\varphi}$, is much larger compared to the typical size of the scattering body d. One can use then a straightforward Taylor expansion (assuming that $k\rho \gg 1$) to express the fundamental Green's function in terms of series of axial multipoles lying at the origin, i.e.

$$\frac{e^{ikR}}{R} = \sum_{l=0}^{\infty}\frac{(-d)^l}{l!}\xi^l\frac{\partial^l}{\partial x^l}\left(\frac{e^{ik\rho}}{\rho}\right). \tag{26}$$

Substitute (26) in (25) and note (see [20], Eq. (3.1.3a)) that the angular SWF can be expanded in terms of Legendre polynomials $P^m_n(\xi)$ as

$$S_{mn}(c,\xi) = {\sum_{r=0,1}^{\infty}}{}'\,\hat{d}^{mn}_r(c)P^m_{m+r}(\xi), \tag{27}$$

where the coefficients $\hat{d}^{mn}_r(c)$ can be determined from employing a simple recursion formula (see [7, 19–23]). Note that the "prime" summation $\sum{}'$ in (27) is taken only over even values of r when $n-m$ is even and only over odd values of r when $n-m$ is odd. The integral in the right hand side of (27) can then be analytically evaluated by making use of the following relation [3]:

$$\begin{aligned}&\int_{-1}^{1}\xi^n\left(1-\xi^2\right)^{m/2}P^m_{m+r}(\xi)\,d\xi\\ &= \frac{2^{m+n+1}(m+n)!(2m+n)!}{(2m+2n+1)!}\delta_{nr}.\end{aligned} \tag{28}$$

Substitution of Eqs. (26)–(28) into (27) finally leads (for $k\rho \gg 1$) to

$$\begin{aligned}&\tilde{B}^m_n(c)S_{mn}(c,\mu)R^{(3)}_{mn}(c,\zeta)e^{im\varphi}\\ &= {\sum_{l=0,1}}{}'\,d^{m+l+1}\frac{2^{m+l+1}(-1)^l(m+l)!(2m+l)!}{l!(2m+l+1)!}\\ &\times\hat{d}^{mn}_l(c)\frac{\partial^l}{\partial x^l}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)^m\left(\frac{e^{ik\rho}}{\rho}\right) = f_{mn}(\theta,\varphi;c)\left(\frac{e^{ik\rho}}{k\rho}\right),\end{aligned} \tag{29}$$

where

$$\begin{aligned}f_{mn}(\theta,\varphi;c) &= 2c{\sum_{l=0,1}}{}'\,\frac{(-1)^l(2ic)^{m+l}(m+l)!(2m+l)!}{l!(2m+l+1)!}\\ &\times\hat{d}^{mn}_l(c)(\cos\theta)^l(\sin\theta)^m e^{im\varphi}\end{aligned} \tag{30}$$

represents the far-field dimensionless amplitude (directivity) or the so-called "angle distribution factor" of the scattered sound field.

It is also worth mentioning that as $k\to 0$ (or $c\to 0$), the Helmholtz equation (1) reduces to the Laplace's equation and thus the kernel in (18) degenerates into

$$\lim_{k\to 0}k^{2m+1}\frac{h^{(1)}_m(kR)}{(kR)^m} \to -i\frac{(2m)!}{2^m m!}\frac{1}{R^{2m+1}}. \tag{31}$$

Furthermore, since $S_{mn}(0,\mu) = P_n^m(\mu)$ and the corresponding external harmonic is simply given by $P_n^m(\mu)Q_n^m(\zeta)e^{im\varphi}$, Eq. (18) clearly renders

$$\frac{(2m)!(-1)^m d^{2m+1}}{2^m m!}\int_{-1}^{1}\frac{\left(1-\xi^2\right)^{m/2}}{R^{2m+1}}P_n^m(\xi)d\xi = \tilde{B}_n^m(0)\tilde{P}_n^m(\mu)\tilde{Q}_n^m(\zeta), \tag{32}$$

where the orthogonality relation for Legendre' polynomials corresponding to (4) is

$$\int_{-1}^{1} P_n^m(\mu)P_{n'}^m(\mu)d\mu = \frac{2\delta_{nn'}}{(2n+1)}\frac{(n+m)!}{(n-m)!}. \tag{33}$$

Let us again take $\mu \to 1$ in (32), which implies that on the axis of symmetry ($y = z = 0$), $R = d(\zeta - \xi)$. The resulting integral in (32) can then be analytically computed for $\zeta >> \xi$ as

$$\int_{-1}^{1}\frac{\left(1-\xi^2\right)^{m/2}}{(\zeta-\xi)^{2m+1}}P_n^m(\xi)d\xi = \frac{2^{n+1}n![(n+m)!]^2}{(2n+1)!(2m)!(n-m)!}\frac{1}{\zeta^{n+m+1}}. \tag{34}$$

Following [21] (Ch. 8) we recall that

$$\tilde{P}_n^m(1) = \frac{(n+m)!}{2^m m!(n-m)!}; \qquad \lim_{\zeta\to\infty}\tilde{Q}_n^m(\zeta) = \frac{(-1)^m 2^n n!(n+m)!}{(2n+1)!\zeta^{n+m+1}}. \tag{35}$$

Finally, substituting (34) and (35) in (32) and letting $\zeta \to \infty$, gives

$$\tilde{B}_n^m(0) = 2, \tag{36}$$

which, by virtue of (25) and for the particular case where $d = 1$ (i.e. when all distances are normalized with respect to half the distance between foci), leads to

$$P_n^m(\mu)Q_n^m(\zeta)e^{im\varphi} = \frac{1}{2}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)^m \int_{-1}^{1}\frac{\left(1-\xi^2\right)^{m/2}P_n^m(\xi)}{\sqrt{\left(x-\xi^2\right)^2 + y^2 + z^2}}d\xi. \tag{37}$$

Eq. (37) coincides with the so-called Havelock's "ultimate image spheroid" theorem [5], and here we provide an independent proof of this useful identity, which was extensively used in [1, 2].

4. ACOUSTIC SCATTERING FROM A SPHEROID NEAR A WALL

In order to demonstrate the preceding methodology, we choose a simple axisymmetric configuration of a rigid prolate spheroid placed near a planar (soft or hard) wall (see Fig. 1a). The surface of the spheroid is given by $\zeta = \zeta_0 > 1$ and $h > a$ denotes the distance of its center from the wall. A general "incident" acoustic wave field near the spheroid preserving axial symmetry ($m = 0$), can be expressed in terms of SWF as

$$\psi_{in}(\mu,\zeta;c,h) = \sum_{n=0}^{\infty} A_n(c,h)S_{0n}(c,\mu)R_{0n}^{(1)}(c,\zeta). \tag{38}$$

The coefficients $A_n(c,h)$ are generally prescribed, where following [20] (Eq. (5.3.3)), $A_n(c,h) = \frac{2}{N_{0n}(c)}\mathrm{Re}\left\{i^n e^{-ich}\right\}$ or $A_n(c,h) = \frac{2}{N_{0n}(c)}\mathrm{Im}\left\{i^n e^{-ich}\right\}$, depending if the wall at $x = h$ is "hard" or "soft" for an incident monochromatic wave-field $e^{ic(x-h)}$. It is important to note that in the sequel we have assumed for simplicity that all distances are normalized with respect to d (half the distance between foci). In a similar manner, if the incident wave-field is generated by an acoustic point source (monopole) at $P(-x_s,0,0)$ lying on the axis of symmetry at a distance $x_s > 1$ from the origin, then according to (3) one gets

$$A_n(c,h;x_s) = (2c/N_{0n})\mathrm{Re}\left\{iS_{0n}(c,-1)R_{0n}^{(3)}(c,x_s) \pm iS_{0n}(c,1)R_{0n}^{(3)}(c,2h+x_s)\right\}. \tag{39}$$

Here the upper (plus) sign corresponds to "hard" (Neumann-type B.C.) wall and the lower (minus) sign to a "soft" wall (Dirichlet-type B.C.).

The scattered wave-field under axisymmetric forcing can be then simply expressed by the present "ultimate" image method (25) as

$$\psi_{\mathrm{scatt}}(\mu,\zeta;c,h) = -\sum_{n=0} C_n(c,h)\frac{\dot{R}_{0n}^{(1)}(c,\zeta_0)}{\dot{R}_{0n}^{(3)}(c,\zeta_0)} \times\left[S_{0n}(c,\mu)R_{0n}^{(3)}(c,\zeta) \pm \frac{1}{\tilde{B}_n^0(c)}\int_{-1}^{1}\frac{e^{ikR'}}{R'}S_{0n}(c,\xi)d\xi\right], \tag{40}$$

where $R' = \left[(x-2h-\xi)^2 + y^2 + z^2\right]^{1/2}$ and the upper/lower signs denote cases of hard/soft wall respectively. Here R' represents the distance from the "image" singularity at $(2h+\xi,0,0)$ and any external field point (x,y,z). For the present axisymmetric case, one can also apply (3) which renders for $|\xi| < 1$,

$$\frac{e^{icR'}}{R'} = 2ic\sum_n (N_{0n})^{-1}S_{0n}(c,\mu) \times R_{0n}^{(1)}(c,\zeta)S_{0n}(c,1)R_{0n}^{(3)}(c,2h+\xi). \tag{41}$$

Explicit expressions for the unknown scattering coefficients $C_n(c,h)$ in (40) can then be obtained by enforcing the Neumann boundary conditions on $\zeta = \zeta_0$ for the total acoustic field $\psi_{\text{total}} = \psi_{\text{in}} + \psi_{\text{scatt}}$ by making use of the orthogonality properties (4) of the SWF. Thus, one finally gets by virtue of (23) the following linear system ($n = 0, 1, 2, \ldots$) for the coefficients C_n:

$$A_n = C_n \pm \sum_{m=0}^{\infty} C_m D_{mn}(c,h), \tag{42}$$

where

$$\begin{aligned} D_{mn}(c,h) &= \frac{S_{0n}(c,1)}{N_{0n}(c)\tilde{R}_{0n}^{(1)}(c,1)} \frac{\dot{R}_{0m}^{(1)}(c,\zeta_0)}{\dot{R}_{0m}^{(3)}(c,\zeta_0)} \\ &\times \int_{-1}^{1} S_{0m}(c,\xi)\dot{R}_{0n}^{(3)}(c,2h+\xi)\,d\xi. \end{aligned} \tag{43}$$

The upper dot in (43) denotes differentiation with respect to the argument ζ (evaluated on $\zeta = \zeta_0$). Note that the case of a "remote" wall $(h \to \infty)$ corresponds to the "unbounded" case where $D_{mn} \to 0$. Furthermore, since the SWF in (43) are tabulated or can be computed by using available subroutines, both A_n and matrix D_{mn} can be considered as prescribed.

Solving the linear system (42) for the coefficients $C_n(c,h)$ is rather straightforward for any acoustic wavelength k, spheroid geometry ζ_0 and wall distance h. For example, rewriting (42) in a matrix form as $\mathbf{A} = \mathbf{C} \pm \mathrm{D}\mathbf{C}$, where D denotes the diagonal symmetric matrix of (43), the unknown coefficient vector $\mathbf{C}$ can be explicitly expressed in terms of the prescribed vector $\mathbf{A}$ by the following successive matrix operation: $\mathbf{C} = \mathbf{A} \mp \mathrm{D}\mathbf{A} \pm \mathrm{DD}\mathbf{A} \mp \mathrm{DDD}\mathbf{A}\ldots$. One should note that as $h \to \infty$, $\mathrm{D} \to 0$ and $\mathbf{C} = \mathbf{A}$. An asymptotic large-spacing $(a/h \ll 1)$ approximation can be obtained from (43) by recalling [20] that $R_{0n}^{(3)}(c,z) \to (cz)^{-1}\exp\{-i[cz - \pi(n+1)/2]\}$ as $z \to \infty$. Thus, by virtue of (22) and (43) one gets the following leading far-field explicit expression for the matrix D:

$$\begin{aligned} D_{mn}(c,h) &= \frac{S_{0m}(c,1)S_{0n}(c,1)}{N_{0n}(c)} \frac{\dot{R}_{0m}^{(1)}(c,\zeta_0)}{\dot{R}_{0m}^{(3)}(c,\zeta_0)} \\ &\times i^{(m+n+1)}e^{-2ich}(a/h) + o(a/h)^2, \end{aligned} \tag{44}$$

which can be combined with the above iterative procedure to render a formal asymptotic expansion for the coefficient A in terms of the small ratio a/h. Fast convergence is expected for large wall-spacing. As a final remark, we note that such a relatively simple solution for the scattering coefficient C can only be made possible by virtue of the newly derived relation (25), which also provides analytic expressions for the far-field sound scattering and directivity amplitude.

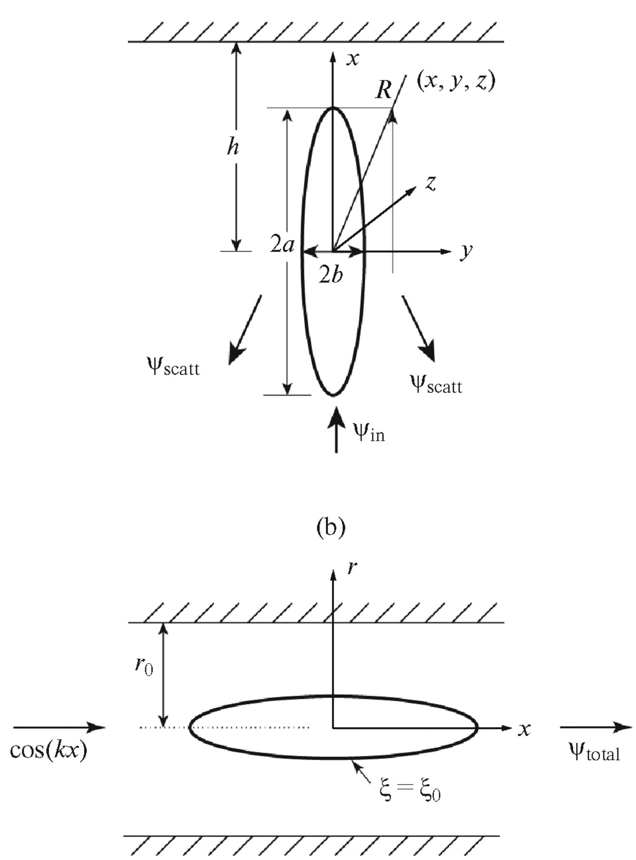

Fig. 1. Definition sketch.

5. ACOUSTIC SCATTERING FROM A SPHEROID INSIDE A CYLINDRICAL DUCT

Yet another example demonstrating the above methodology is sound scattering from a soft spheroid placed on the axis of a rigid cylindrical duct of radius r_0 (Fig. 1b) excited by an axisymmetric acoustic wave $\cos(kx)$, where following [20] (Eq. (5.3.2))

$$\cos(kx) = \sum_n \frac{i^n}{N_{0n}} S_{0n}(c,1)\,\mathrm{S}_{0n}(c,\mu)\,\mathrm{R}_{0n}^{(1)}(c,\zeta), \quad c = kd. \tag{45}$$

Making use of the following integral relation for the Green's function of the Helmholtz equation [7] involving the modified Bessel function of the second kind:

$$\frac{e^{ikR}}{R} = \frac{2}{\pi}\int_0^{\infty} K_0\left[r\sqrt{k^2+\alpha^2}\right]\cos(\alpha x)\,d\alpha, \tag{46}$$

enable us to express the total acoustic field within the duct in terms of SWF as

$$\psi_{\text{total}} = \sum_n \frac{i^n}{N_{0n}} S_{0n}(c,1)\{S_{0n}(c,\mu)\,\mathrm{R}_{0n}^{(1)}(c,\zeta) + C_n(c,r_0)\left[S_{0n}(c,\mu)R_{0n}^{(3)}(c,\zeta) + H_n(r,x)\right]\}, \tag{47}$$

where $C_n(c,r_0)$ denote the unknown amplitude coefficients of the scattering wave field and $H_n(r,x)$ is a harmonic function satisfying the Laplace's equation to be determined. We use here concurrently both spherical (R,μ) as well as cylindrical (r,x) coordinates systems preserving axial symmetry.

Employing next the general theorem given in (24), (25) together with (46), one gets

$$H_n(r,x) = \frac{2}{\pi\tilde{B}_n^0(c)}\int_0^\infty\int_{-1}^1 \frac{\sqrt{k^2+\alpha^2}}{\alpha}\frac{K_1\left[r_0\sqrt{k^2+\alpha^2}\right]}{I_1(r_0\alpha)} \times I_0(r\alpha)S_{0n}(c,\xi)\cos\alpha(x-\xi)\,d\xi\, d\alpha, \tag{48}$$

where according to (24) $\tilde{B}_n^0(c) = 2ic\widetilde{\dot{R}}_{0n}^{(1)}(c,1)$ and $I_n(z)$ is the modified Bessel function of the first kind.

The scattering wave field in (47), (48) satisfies the Neumann boundary condition $\frac{\partial\psi_{\text{total}}}{\partial r} = 0$ on the cylindrical duct $r = r_0$. The unknown coefficient in (47) is finally found by imposing the Dirichlet condition $\psi_{\text{total}} = 0$ on the "soft" $\zeta = \zeta_0$ spheroid. Thus, we get the following linear system of equations for the coefficients C_n:

$$\dot{R}_{0n}^{(1)}(c,\zeta_0) + C_n(c,r_0)\dot{R}_{0n}^{(3)}(c,\zeta_0) + \sum_m F_{mn}C_m(c,r_0) = 0, \tag{49}$$

where according to (48) the matrix F_{mn} in (49) is explicitly given by

$$F_{mn} = \frac{2}{\pi\tilde{B}_n^0(c)N_{0n}}\int_0^\infty\int_{-1}^1\int_{-1}^1 \frac{\sqrt{k^2+\alpha^2}}{\alpha}\frac{K_1\left[r_0\sqrt{k^2+\alpha^2}\right]}{I_1(r_0\alpha)} \times I_0\left[\alpha(\zeta_0^2-1)^{1/2}(1-\mu^2)^{1/2}\right] \times S_{0m}(c,\xi)\cos(\alpha\mu\zeta_0)\cos(\alpha\xi)]d\mu\, d\xi\, d\alpha. \tag{50}$$

The amplitude of the scattering field for the "unbounded" case (no duct) is readily found from (49) as $C_n(c,\infty) = -\dot{R}_{0n}^{(1)}(c,\zeta_0)/\dot{R}_{0n}^{(3)}(c,\zeta_0)$ since $F_{mn} \to 0$ as $r_0/a \to \infty$ (infinitely large blockage) and a correction due to finite blockage can be semi-analytically found by solving the linear system (49). In a similar manner to the previous case (44), one can also find an asymptotic (large-spacing) expression by letting $r_0/a \gg 1$ in (50) which to leading-order in (a/r_0) yields $C_n(c,r_0) = C_n(c,\infty) + \sum_m \bar{F}_{mn}(c,r_0)C_m(c,\infty)$, where

$$\bar{F}_{mn}(c,r_0) = \frac{8S_{0m}(c,1)}{\tilde{B}_n^0(c)N_{0n}\dot{R}_{0n}^{(3)}(c,\zeta_0)}\mathrm{Re}\{i^{-m}\}\int_0^\infty\left(\frac{\sqrt{k^2+\alpha^2}}{\alpha}\right)^{1/2} \times \frac{\sin\alpha}{\alpha}\tilde{R}_{om}^{(1)}(c,\alpha)e^{-r_0(\alpha+\sqrt{k^2+\alpha^2})}d\alpha. \tag{51}$$

It is finally noted that such explicit expressions for the scattering acoustic coefficients of a spheroidal shape lying in a duct can be obtained only by virtue of the newly derived integro-differential relation (25) for the SWF eigensolutions.

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