

Strong and Weak Nonlinear Dynamics: Models, Classification, Examples¹

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Abstract—The difference between strong and weak nonlinear systems is discussed. A classification of strong nonlinearities is given. It is based on the divergence or inanity of series expansions of the equation of state commonly used in the study of weak nonlinear phenomena. Such power or functional series cannot be used in three cases: (i) if the equation of state contains a singularity; (ii) if the series diverges for strong disturbances; (iii) if the linear term is absent, and higher nonlinearity dominates. Strong nonlinearities are known in acoustics, optics, mechanics and in quantum field theory. Mathematical models, solutions and observed phenomena are presented. For example, an equation of Heisenberg type and its generalization for strongly nonlinear wave system are given. In particular, exact solutions of new “quadratically cubic” Burgers and Riemann–Hopf equations are discovered.

Keywords: strong nonlinearity, new models, singularity, Burgers, Heisenberg, Riemann–Hopf equations

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INTRODUCTION

A modern trend in physics is the increasing interest in strongly nonlinear wave (SNW) dynamics. It is generated by the physics of high density of energy, extreme states of matter, and astrophysics, as well as by new experiments in laser physics and explosive waves connected with high energy localization.

Up to now, mainly weak waves have been studied. Even weakly nonlinear waves (WNW) can demonstrate strongly displayed nonlinear phenomena. For example, nonlinear optics deals with laser fields whose strength is much smaller than the intra-atomic field:

$E \ll E_{AT} \sim e/r_b^2 \sim 10^{11} \text{ V/m}$ (here r_b is the Bohr radius).

Nevertheless, effects of weak nonlinearity accumulate with time (within many periods of vibrations) or with distance traveled (within many wavelengths) and initiate strongly expressed effects. For instance, the wave energy can almost completely spill over from one frequency wave to another (second harmonic generation, parametric amplification). Formation of solitons and self-focusing of beams are also results of weak nonlinearity.

In strong fields, $E \sim E_{AT}$, nonlinear transformations happen rapidly over a short time, or within a

short distance. Irreversible changes take place in the medium right up to its destruction. But even high-intensity laser waves will be weak at propagation in vacuum. Only a wave with the strength $E \sim E_{vac} \sim m^2 c^3 / e \hbar \sim 10^{18} \text{ V/m}$, is strong here because it can alter the vacuum itself and create electron–positron pairs.

By analogy, one may distinguish between SNW and WNW in acoustics. When a shock front appears at a distance of 10^2 – 10^3 wavelengths in a liquid, nonlinearity is weak but strongly expressed. The acoustic pressure is here 10^5 – 10^6 Pa , much less than the internal pressure of 10^9 Pa . Typical strongly expressed effects of weak acoustic nonlinearity are: the transformation of a smooth single pulse to a triangular looking profile with a leading shock front; the transformation of an initially harmonic wave to a saw-tooth-shaped wave containing one shock per period.

A different case occurs if there are impurities (cavitation embryos) in water. The cavitation resistance falls down, and the acoustic wave can at low pressures break the liquid and create bubbles. An explosive wave is also strongly nonlinear. It destroys the medium. At nuclear explosions even new chemical elements can appear. Extreme states of matter are realized in white dwarfs, neutron stars and in black holes as result of gravitation collapse and accretion. A wave initiating an

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inert-less phase transition can also be considered as a SNW.

It is necessary to answer the question: when the wave is said to be weak or strong—it is in comparison with what? If the wave or its parameters (spectrum, shape, other characteristics) is compared with its own at the initial moment of time, and a big difference in the parameters is observed, one can point to a WNW. If the wave field is compared with a typical magnitude of the same field inside the medium and these magnitudes are close one to another, one can point to a SNW.

In the study of WNW, the equation of state (or determining equation) can be expanded in a power or functional series. The expansion of the adiabatic equation of state in powers of density ρ' and pressure p' disturbances in the vicinity of equilibrium state (p_0, ρ_0) can be one example:

$$p' = p_0 \left(\frac{\rho_0 + \rho'}{\rho_0} \right)^\gamma = p_0 + c_0^2 \rho_0 \left[\frac{\rho'}{\rho_0} + \frac{\gamma-1}{2} \left(\frac{\rho'}{\rho_0} \right)^2 + \frac{(\gamma-1)(\gamma-2)}{6} \left(\frac{\rho'}{\rho_0} \right)^3 + \dots \right]. \quad (1)$$

Here $\gamma = c_p/c_v$ is the ratio of the thermal capacities, $c_0^2 = \gamma p_0/\rho_0$ is the square of the sound velocity. The expansion (1) in powers of the small quantity ρ'/ρ_0 is commonly used in nonlinear acoustics [1]. The second term inside the brackets (1) is related to quadratic nonlinearity, while the third term is related to a cubic one.

Another expansion of polarization vector in powers of ratio of external electric and intra-atomic field is used in nonlinear optics [2]:

$$\begin{aligned} \vec{P} = & \int_0^\infty \hat{\kappa}(\tau) \vec{E}(t-\tau) d\tau + \iint_0^\infty \hat{\chi}^{(2)}(\tau_1, \tau_2) \\ & \times \vec{E}(t-\tau_1) \vec{E}(t-\tau_1-\tau_2) d\tau_1 d\tau_2 + \iiint_0^\infty \hat{\chi}^{(3)}(\tau_1, \tau_2, \tau_3) \\ & \times \vec{E}(t-\tau_1) \vec{E}(t-\tau_1-\tau_2) \vec{E}(t-\tau_1-\tau_2-\tau_3) d\tau_1 d\tau_2 d\tau_3 \end{aligned} \quad (2)$$

Here $\hat{\kappa}(\tau)$ is the tensor of linear polarization, and $\hat{\chi}^{(2)}, \hat{\chi}^{(3)}$ are tensors of quadratic and cubic nonlinearities.

A similar Frechet representation of the series by a multiple integral generalizing the Taylor series was used as far back as by Volterra in his theory of hereditary elasticity [3].

However, such expansions cannot be used in three cases. First, if the determining equation $p(\rho), \vec{P}(\vec{E})$ contains a singularity, like for “clapping” and for Hertz nonlinearities of heterogeneous solids [4], as well as for nonlinearity of vibrant-impact systems [5].

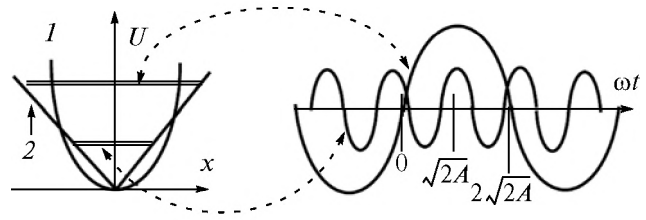


Fig. 1. Vibration in potential well with singularity at $x = 0$.

Secondly, if the series expansions (1), (2) are divergent for very strong fields [6]. Thirdly, if the linear term is absent and higher nonlinearities dominate. Such nonlinearities are known in quantum field theory [7] and in mechanics [8].

Examples are given below of nonlinear vibration and wave systems of the first, second and third kinds.

EXAMPLES OF 3 TYPES OF VIBRATION SYSTEMS

A. Example of Strong Nonlinearity of the 1st Type: there is a Singularity

Let us consider an anharmonic oscillator with a “module-like” potential:

$$m \frac{d^2 x}{dt^2} = F = -\frac{\partial U}{\partial x}, \quad U = -kx_0|x|, \quad \omega_0^2 = \frac{k}{m}x_0. \quad (3)$$

The “module-like” well (curve 2 in Fig. 1) is shown by two rays. It is similar to the parabolic well of a harmonic oscillator (curve 1). The solution here is even simpler than that to the linear problem. It can be described by simple quadratic functions of time:

$$x = A - \frac{1}{2}(\omega_0 t - \sqrt{2A})^2, \quad 0 < \omega_0 t < 2\sqrt{2A}, \quad (4)$$

$$x = \frac{1}{2}(\omega_0 t + \sqrt{2A})^2 - A, \quad -2\sqrt{2A} < \omega_0 t < 0.$$

The spectral expansion of this solution is given by the series:

$$x = \frac{32}{\pi^3} A \sum_{m=0}^{\infty} (2m+1)^{-3} \sin \left[\frac{\pi}{2} (2m+1) \frac{\omega_0 t}{\sqrt{2A}} \right]. \quad (5)$$

There are only odd harmonics, whose amplitudes are $\sim A$. The period increases as $\sqrt{2A}$. With decrease in amplitude A the point mass goes down to the bottom of well, and approaches the singular point. Consequently, there is no limiting transition to the linear problem at small amplitudes when $A \rightarrow 0$.

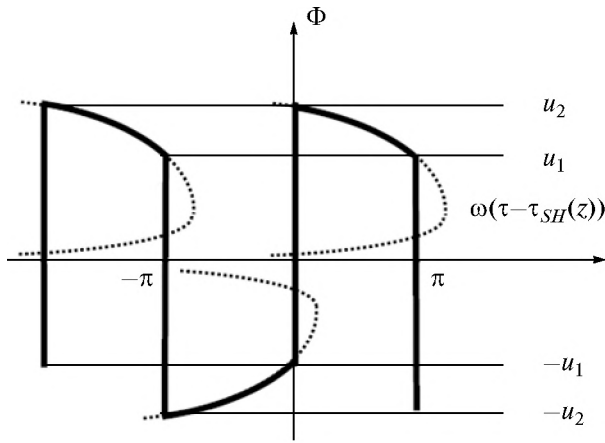


Fig. 2. Trapezoidal "saw" in quadratically cubic medium.

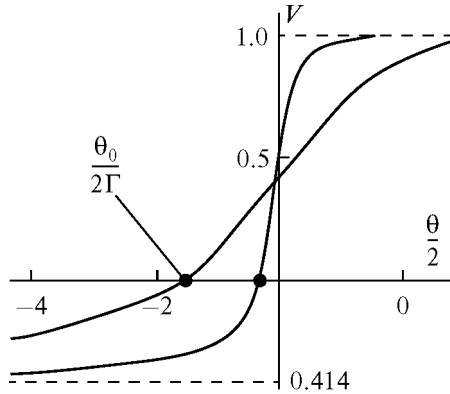


Fig. 3. Stable shock front of compression in a quadratically cubic medium for $\Gamma = 1; 0.2$.

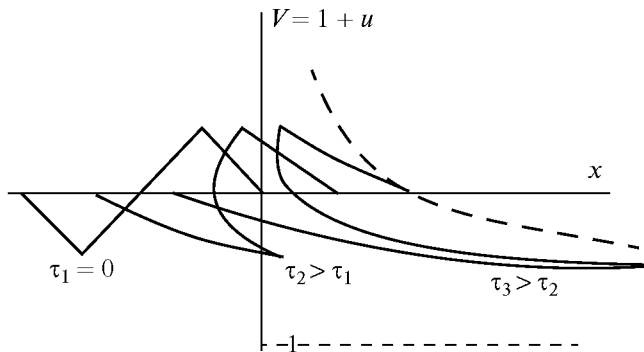


Fig. 4. Distortion of SNW described by the model (18), (19).

B. Example of Strong Nonlinearity of the 2nd Type: the Divergence of Series for Strong Vibrations

Consider the anharmonic oscillator

$$\frac{d^2x}{dt^2} + \frac{\omega_0^2 x}{\sqrt{1 - x^2/a^2}} = 0. \quad (6)$$

The singularity appears here at $|x| \rightarrow a$. At small displacements $|x| \ll a$, the behavior is like that of a harmonic oscillator. The energy integral is:

$$E = \frac{1}{2} \left(\frac{dX}{d\tau} \right)^2 + (1 - \sqrt{1 - X^2}), \quad X = \frac{x}{a}, \quad \tau = \omega_0 t. \quad (7)$$

At $|X| \ll 1$, the energy takes a simple quadratic form. At large displacements $|X| > 1$, the vibrating mass "disappears" in the space of real coordinates and moments.

C. Example of Strong Nonlinearity of the 3rd Type: there is No Linear Term in the Series Expansion

Let us consider the Duffing equation:

$$\frac{d^2x}{dt^2} + \omega_0^2 x \left(1 + \frac{x^2}{a^2} \right) = 0, \quad (8)$$

where the linear term (the unity) in parenthesis is omitted. This equation

$$\frac{d^2X}{d\tau^2} + X^3 = 0, \quad X = \frac{x}{a}, \quad \tau = \omega_0 t \quad (9)$$

was considered before in quantum [7] and classical [8] physics. Its solution can be expressed through Jacobi elliptic functions. The solution and Fourier expansion of the odd harmonics are:

$$\frac{x}{a} = A \operatorname{sd} \left(A \omega_0 t \middle| \frac{1}{2} \right), \quad T = \frac{\Gamma^2(1/4)}{\sqrt{\pi}} \frac{1}{\omega_0 A}. \quad (10)$$

$$\begin{aligned} \frac{x}{a} = & A \frac{16\pi^{3/2}}{\Gamma^2(1/4)} \sum_{m=1}^{\infty} (-1)^{m+1} \\ & \times \frac{\exp(-\pi(m-1/2))}{1 + \exp(-2\pi(m-1/2))} \sin \left[(2m-1) \frac{2\pi^{3/2}}{\Gamma^2(1/4)} A \omega_0 t \right]. \end{aligned} \quad (11)$$

The models listed above describe real systems. The 1st type oscillator (3) describes vibrations of the force field, which is homogeneous in both half-spaces, but changes its direction at the transition through $x = 0$. Such motion is realized near a plane-parallel plate creating homogeneous gravity. If a small orifice is drilled in the plate, the point mass will be attracted to the other side of plate when passing the orifice.

The system of 2nd type (6) has a potential well in the region $-1 < X < 1$. If the well is restricted by reflecting walls, the motion will be concentrated inside this area. If the potential is put to zero at $|X| > 1$, two barriers appear at the well edges. A particle with large enough energy really leaves the well and "disappears".

The simplest system of the 3rd kind is shown in Fig. 5. There is a constraint: the mass can slide only along a guide bar. The spring is linear. It is unstretched in equilibrium. The equation of motion of the mass coincides with Eq. (9).

EXAMPLES OF THREE TYPES OF WAVE SYSTEMS

A. Wave System of the 1st Type: there is a Singularity

A cubic nonlinear medium for strong waves has the following equation of state: $u/u_0 = \beta(p/p_0)^3$. Let u, p be the vibration velocity and pressure. It is convenient to approximate this cubic function by two branches of a quadratic parabola: $u/u_0 = \beta|p/p_0|(p/p_0)$. In many cases it is adequate for real wave systems, and sometimes this model is convenient for qualitative physical analysis.

Plane waves in such “quadratically cubic” systems are described by the equation of Riemann–Hopf type. The solution for the given initial profile $\Phi(\tau)$ at the border $z = 0$ of the medium is determined by the implicit function:

$$\frac{\partial u}{\partial z} = \frac{\beta}{c_0^2} |u| \frac{\partial u}{\partial \tau} \rightarrow u = \Phi\left(\tau + \frac{\beta}{c_0^2} |u| z\right). \quad (12)$$

For an initial harmonic wave, its spectrum contains a Fourier expansion in odd harmonics:

$$u = u_0 \sum_{n=1}^{\infty} C_n(Z) \sin(n\omega\tau + \varphi_n(Z)), \quad (13)$$

$$Z = \frac{z}{z_{SH}} = \frac{\beta}{c_0^2} \omega u_0 z.$$

$$C_n = [1 - (-1)^n] \frac{2}{nZ} \times \left[\left(\frac{2}{\pi n} - E_n(nZ) \right)^2 + J_n^2(nZ) \right]^{1/2},$$

$$\tan \varphi_n = \frac{(2/\pi n) - E_n(nZ)}{J_n(nZ)}.$$

Here J_n, E_n are the Bessel and Weber functions.

It is interesting to construct the solution to (12) for the distances where the wave contains shocks. Looking for a solution in the form $u = A(z)\Phi(\tau - \tau_{SH}(z))$, where the function $\tau_{SH}(z)$ describes the motion of shock, and the function $\Phi(\tau)$ the quasi-stable shapes of smooth sections of wave profile, we find:

$$A(z) = \frac{u_0}{1 + z/z_0}, \quad \omega(\tau - \tau_{SH}(z)) + C$$

$$= \ln|\Phi| + Z_0|\Phi|, \quad \omega\tau_{SH}(z) = -\ln\left(1 + \frac{z}{z_0}\right), \quad (14)$$

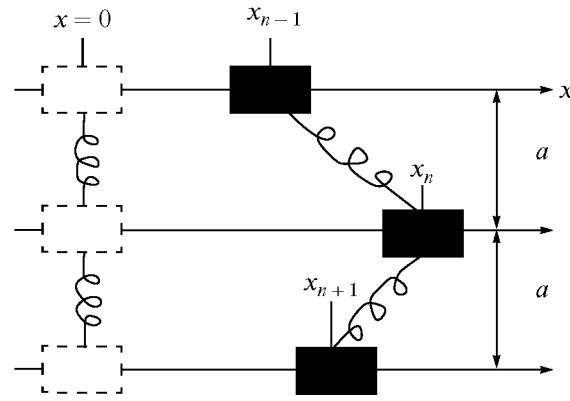


Fig. 5. Chain of masses demonstrating strongly nonlinear behavior.

here $Z_0 = z_0/z_{SH} = (\beta/c_0^2)\omega u_0 z_0$. By placing the shocks in the profile, we construct the periodic wave shown in Fig. 2. The “teeth” have a trapezoidal form, as distinct from the usual triangular “saw” in quadratic medium [1, 9]. Each period contains two shocks: one compression and one rarefaction. The last type does not exist in quadratic media.

To describe the shock structure, the quadratically cubic analogue of Burgers equation must be solved:

$$\frac{\partial V}{\partial Z} = \frac{1}{2} \frac{\partial}{\partial \theta} (|V|V) + \Gamma \frac{\partial^2 V}{\partial \theta^2}, \quad (15)$$

$$V = \frac{u}{u_0}, \quad \theta = \omega\tau, \quad Z = \frac{z}{z_{SH}}.$$

This remarkable equation can be reduced to a linear form and solved exactly.

One of the principal exact solutions describes a stable compression shock:

$$V = \alpha^2 \left(\frac{\theta_* - \theta_0}{2\Gamma} \right) \left[1 - \alpha \left(\frac{\theta_* - \theta_0}{2\Gamma} \right) \right]^{-1},$$

$$-\infty < \theta_* < \theta; \quad V = \alpha \left[1 + \sqrt{2} \tanh\left(\alpha \sqrt{2} \frac{\theta_*}{2\Gamma}\right) \right], \quad (16)$$

$$\theta_0 < \theta_* < \infty.$$

Here: $\alpha \equiv (\sqrt{2} - 1)$, $\theta_0/2\Gamma = -(\alpha\sqrt{2})^{-1} \times \arctan(\sqrt{2}/2)$, $\theta_* = \omega(\tau - \tau_{SH}(z))$.

The shape of shock front (16) is shown in Fig.3 for two different values of the parameter Γ . By analogy, the rarefaction front can be calculated.

B. Wave System of the 2nd Type: there is a Divergence for Strong Disturbances

One system of such type is well-known [9]. It is the Earnshaw equation describing (in Lagrangian representation) the 1D motion of a compressive gas:

$$\frac{\partial^2 \xi}{\partial t^2} = c_0^2 \left(1 + \frac{\partial \xi}{\partial x}\right)^{-(\gamma+1)} \frac{\partial^2 \xi}{\partial x^2}. \quad (17)$$

Here ξ is the displacement. The weak nonlinearity corresponds to a small Mach number, or small strain $|\partial \xi / \partial x| \ll 1$. If only the main quadratic nonlinearity is kept, the simplified equation takes into account only weak nonlinearity but can describe strongly expressed accumulated nonlinear phenomena which are studied well.

Therefore, it is interesting to draw attention to SNW, when strain is comparable with unity. One can see, that at large negative strain, $\partial \xi / \partial x < -1$, a singularity appears in equation (17). It corresponds to a medium discontinuity. Real fluids contain impurities and breaks at much weaker rarefactions.

Equation (17) can be solved analytically. For simplicity it can be rewritten as

$$\frac{\partial^2 V}{\partial \tau^2} = \frac{\partial}{\partial x} \left(\frac{1}{V^{\gamma+1}} \frac{\partial V}{\partial x} \right), \quad V \equiv 1 + \frac{\partial \xi}{\partial x}, \quad \tau \equiv c_0 t. \quad (18)$$

One can check that the solution of the following 1st order equation

$$\frac{\partial V}{\partial \tau} + \frac{1}{V^\varepsilon} \frac{\partial V}{\partial x} = 0, \quad \varepsilon = \frac{\gamma+1}{2}, \quad (19)$$

will also satisfy (18). In particular, one solution is the implicit function: $V = \Phi(x - V^{-\varepsilon} \tau)$. The nonlinear distortion of an initial wave with 2 subsequent triangular pulses is shown in Fig. 4.

The singularity appears quickly at $\partial \xi / \partial x \rightarrow -1$, $V \rightarrow 0$, when the medium breaks.

Equation (19) has many solutions with singularities. One of the stationary solutions is: $V = C(x - x_0)^{-1/\gamma}$.

Singular solutions are known also for “standard equations” in the theory of WNW. For example, the usual Burgers equation has such singular solutions:

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = \Gamma \frac{\partial^2 V}{\partial x^2} \rightarrow V = -\frac{2\Gamma}{x - x_0}, \quad (20)$$

$$V = -\tan \frac{x - x_0}{2\Gamma}.$$

The behavior of singularities shows particle-like properties. They are stable and can interact with each other. Korteweg–de Vries and many other well-known equations have similar solutions. It is often supposed that singular solutions have no physical meaning. Nevertheless, it is interesting to understand how a singularity forms and what phenomena that can cancel it.

C. Wave System of the 3rd Type: there is No Linear Term

Consider a chain of masses. Each moves along a parallel bar placed at the same distance a from one another (see Fig. 5). It is the generalization of the single oscillator (9) for spatially distributed system [8]. If in equilibrium state all springs are un-tensed, the lin-

ear regime does not exist. Nonlinearity completely determines the motion even at infinitesimally small amplitudes. Such vibrations can evidently be considered as strongly nonlinear, and analytical perturbation methods based on proximity to linear system are not applicable. One needs exact or numerical solutions. Because the stiffness coefficient k of the spring is a constant, the “physical” nonlinearity is absent here. This nonlinearity can be named as “geometrical”, by analogy with nonlinear acoustics.

The equation of motion of a mass of number n in the chain is:

$$\frac{d^2 x_n}{dt^2} = -\frac{k}{2a} [(x_n - x_{n-1})^3 - (x_{n+1} - x_n)^3]. \quad (21)$$

In the continuum approximation, putting $x_n = x(z)$, $x_{n+1} = x(z+a)$, $x_{n-1} = x(z-a)$, $a \ll \lambda$, one can reduce the differential–difference equation (21) to a nonlinear partial differential equation:

$$\frac{\partial^2 \zeta}{\partial t^2} = \frac{\beta}{3} \frac{\partial^2 \zeta^3}{\partial z^2}, \quad \zeta = \frac{\partial x}{\partial z}, \quad \beta = \frac{3k}{2m} a^2. \quad (22)$$

Interestingly, equation (22) has a solution describing standing waves, but traveling waves appear only for springs which are tensed in equilibrium. The propagation velocity increases with increase in tension.

Equation (22) has remarkable properties. First, it permits the separation of variables. Second, it can be reduced to a linear form using new independent variables:

$$\eta = \frac{\partial x}{\partial t}, \quad \zeta = \frac{\partial x}{\partial z}; \quad t = T(\eta, \zeta), \quad z = Z(\eta, \zeta). \quad (23)$$

The corresponding linear equation reads:

$$\frac{\partial^2 T}{\partial \zeta^2} = \beta \zeta^2 \frac{\partial^2 T}{\partial \eta^2}. \quad (24)$$

Third, its solutions can be found by solving the 1st—order equation:

$$\frac{\partial \zeta}{\partial t} = \pm \sqrt{\beta} |\zeta| \frac{\partial \zeta}{\partial z}, \quad (25)$$

which differs only in notations from equation (12) considered above.

When testing experimentally the strongly nonlinear vibrations of the chain in Fig. 5, the friction between masses and guiding bars must be minimized. This was performed through a chain of rotating disks which are described by the same equations [8]. The springs are, as before, governed by the linear Hooke’s law. A photo of the disks and corresponding scheme are shown in Fig. 6.

Experiments with the chain shown in Fig. 6 demonstrated some peculiar properties of strongly nonlinear dynamics [8].

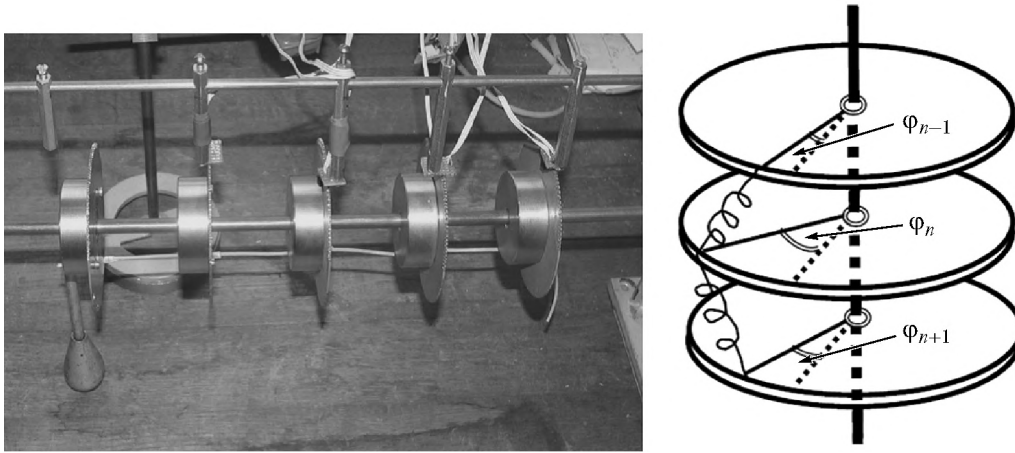


Fig. 6. The chain of disks performing strongly nonlinear torsion vibrations.

STRONG NONLINEARITY AND LOCALIZATION

SNW is associated with localization of energy. This is in turn connected to spatial and temporal focusing, collapse and cumulation.

Significant localization happens when a steam/gas bubble in a liquid collapses. The light radiated by the air bubble collapsing in water under influence of sound (sonoluminescence) was discovered in 1934 [10]. The sonoluminescence of a single bubble in the focal area of an ultrasonic transducer was observed in 1992 [11]. According to estimations, the temperature of plasma in bubbles was of order of 25000 K. By increasing the initial radius of the bubble and reaching the ideal spherical symmetry of collapse, it is possible to achieve an even greater increase in temperature and pressure. However, the true limit of such localization is still unknown [12, 13].

Increase in energy density usually happens in stages. The growth at each stage is limited by nonlinear saturation or by instability. This idea was formulated by Zababakhin as “Hypothesis of instability of cumulation” (1965): “Each unlimited cumulation is unstable; instability not only modifies it, but eliminates it completely (is transformed to limited cumulation)...This is an intuitive idea” [13].

Consider, for example, how one can accumulate high density of energy and create SNW in an acoustic resonator. In the first stage it is necessary to increase the Q -factor of the resonator. Because Q is the ratio of the amplitude of vibration inside the resonator and the amplitude of vibration of the external source of energy, at $Q \gg 1$ the internal vibration can be amplified many times. The “World record” now equals approximately $Q \sim 10^9$ [14]. At $Q \gg 1$ nonlinear phenomena are more pronounced. However, nonlinear steepening leads to shock formation and to nonlinear absorption. Consequently, the Q -factor falls down dramatically.

At the second stage, the shock wave formation has to be suppressed in order to increase the energy. At least 3 methods are known for that [14]. (i) One can artificially introduce phase shifts between harmonics. It is possible, for example, to make a wall with a frequency dependent reflection coefficient. Discrepancies between the phased harmonics forming the shock front disperse the front and decrease the nonlinear losses. (ii) Phase shifts can be introduced using a complicated shape of the cavity. It can be conical, bulbous or similar. In gas-filled cavities, an acoustic pressure of several atmospheres was created. (iii) The third method uses selective absorbers for the 2nd harmonics. In one experiment, one wall was transparent for the 2nd harmonics but reflected the fundamental frequency wave. Shocks could not form and the Q -factor increased.

Consequently, several methods exist to stop both linear and nonlinear losses and increase the Q -factor and internal energy.

The third stage is next. When the shock front formation has been stopped, the boundary nonlinearity caused by the displacement of a movable wall starts to play a determining role. To continue to pump energy into the resonator, one has to modify the vibration of the boundary. At weakly expressed nonlinearity the law must be harmonic, but at strongly expressed nonlinearity the wall must perform short and phased “jerks”.

The fourth stage. If we have been able to overcome both shock formation and boundary nonlinearity, we could meet a new nonlinear phenomena at further pumping of energy, or the medium will be destroyed. A more detailed description of high-power waves in resonators is given in [14], Ch.11.

Corresponding stages can be found in cubic resonators, at strong focusing of wave beams, at forming of extremely strong laser fields, and in other problems connected with SNW.

CONCLUSIONS

Results given above were preliminarily discussed during Scientific school “Nonlinear Waves—2012” (Nizhni Novgorod [15]) and International Congress on Acoustics (Montreal, Canada [16]). These results are connected with the problem of quadratically cubic nonlinear systems described in review [17]. Following conclusions can be drawn:

—It is expedient to distinguish between strongly nonlinear waves (SNW) and waves with strongly expressed weak nonlinearity (WNW).

—The SNW have been studied much less than WNW.

—It is interesting to analyze new nonlinear models and the new physical phenomena of strong wave fields.

—The creation of SNW is connected with the problem of nonlinear localization of energy. The increase in energy and matter density during localization usually consists of several stages bounded by nonlinear saturation or by the development of an instability.

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