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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

Sound Propagation in Two-Dimensional Waveguide with Circular Wavefront[1]

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Abstract—Sound propagation through a waveguide is generally modeled by the Webster horn equation which assumes a planar pressure wavefront. However, most of the sources are non-planar in nature. In this work, a 1-D model is derived for sound propagation through a 2-D waveguide with circular wavefront. The model is derived from the 2-D Helmholtz equation using the weighted residual method. The model assumes a uniform pressure across the angular coordinate at a given radial distance. A 2-D finite element model is used to validate the results for different waveguide geometries and it shows good agreement.

Keywords: Webster horn equation, radial waveguide, numerical method, circular wavefront
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1. INTRODUCTION

Sound propagation in two dimension from a harmonic sound source is given by the Helmholtz equation [1]:

$$\nabla^2 p + \frac{\omega^2}{c^2} p = 0, \tag{1}$$

where p is the pressure amplitude, ω is the angular frequency and c is the speed of sound. The above equation reduces to the Webster horn equation or Webster equation (Eq. (2)), by assuming a uniform pressure across the vertical cross-sectional area of the waveguide [2]. This assumption simplifies the 2-D problem to a 1-D model governed by an ordinary differential equation:

$$\frac{d^2p}{dx^2} + \frac{dp}{dx}\left(\frac{S'}{S}\right) + \frac{\omega^2}{c^2} p = 0, \tag{2}$$

where S is the variable cross section area and *is* the first derivative of S' with respect to x, p is the pressure and ω is the angular frequency.

A detailed review on the Webster horn equation and its solution was given by Eisner [3]. There have been some improvements over the Webster equation. Martin [4] has obtained a hierarchy of one dimensional ordinary differential equations for an axis-symmetric waveguide. The equations were obtained by solving the Helmholtz equation using the power-series expansion method in a stretched radial coordinate. The lowest approximation leads to the Webster equation and second approximation leads to a fourth order differential equation. Rienstra [5] also mentions about sound propagation in a waveguide with mean flow, duct with acoustic lining etc. Webster horn equation has many applications in predicting sound propagation through waveguides, horns, musical instruments etc. Recently we have used Webster horn equation for predicting frequency band structure in one dimensional sonic crystal [6-9] which can be used for the purpose of sound attenuation in selective bands of frequency.

Webster horn equation is also studied [10] for nonlinear acoustic waves in lossy channels with variable cross-sections. The method is based on studying the point symmetry groups for certain types of cross-sectional profiles. Webster horn equation has also been used in predicting acoustic resonance in turbine centrifugal pumps [11]. A transmission line model based on the Webster horn equation is also proposed [12] to model the sound propagation in vocal tract by approximating it as a series of conical horns.

Most of the sources in nature, especially those at high frequencies, have a non-planar pressure profile. Therefore, it is important to account for the non-planar pressure profile and its effect on sound propagation through a waveguide. In this paper, we propose a model similar to the Webster horn equation, but for a wave with cylindrical wavefront (circular in 2-D) propagating through the waveguide as shown in Fig. 1b. The figure illustrates the difference between wave propagating with a plane wavefront versus wave propagating with circular wavefront. Sound wave propagation with planar wavefront is modeled by the standard Webster horn equation (Eq. (2)). The aim of

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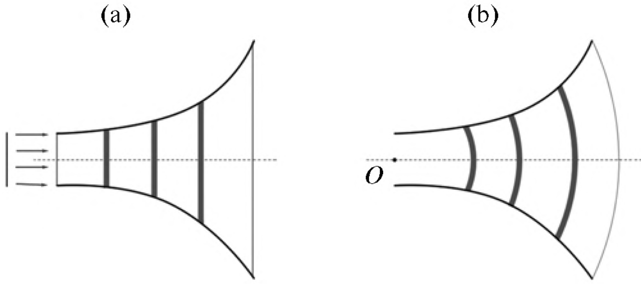

Fig. 1. Sound propagation through a waveguide. Sound wave is modeled with (a) planar wavefront (b) circular wavefront.

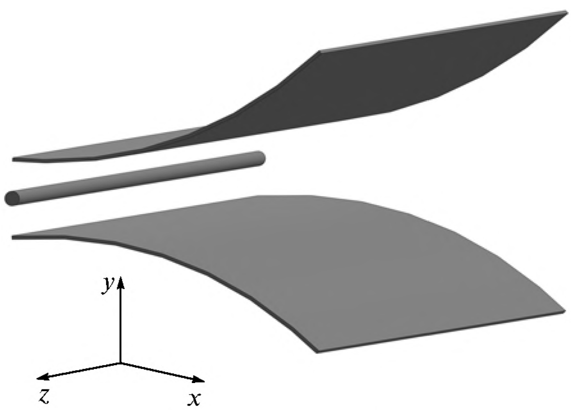

Fig. 2. Sound propagating from a line source through a waveguide. The source and waveguide are long in the z direction so that the analysis is restricted to the 2-D xy plane.

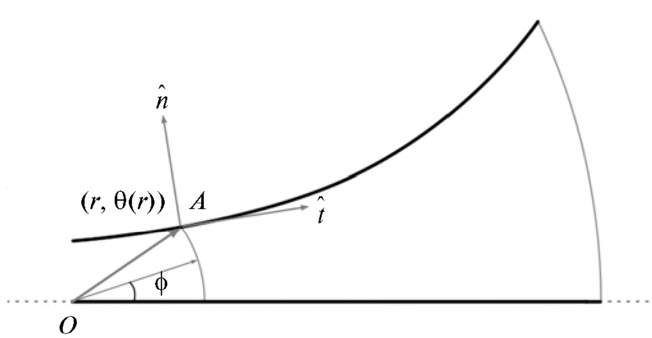

Fig. 3. The symmetric portion of a general waveguide. The figure shows the geometric location of an arbitrary point A in the polar coordinates. The unit normal and tangential vectors at that point are also shown.

the paper is to obtain a governing differential equation similar to the Webster horn equation but for sound wave propagating with a circular wavefront. Effectively, the model has the advantage of reducing a two dimensional problem to a one dimensional model represented by an ordinary differential equation.

The equation obtained is used to evaluate pressure field for different geometries of the waveguide and the results are validated with 2-D finite element simulations. The results from the differential equation match exactly with the finite element simulation for a uniform waveguide. It is shown that for such waveguide in form of sector of circle (with no curvature) the differential equation reduces to the Bessel's equation of zero order. Hence it gives exact solution. However, as the waveguide is made non-uniform by introducing perturbation (semicircle) in the waveguide, the accuracy of 1-D model depends upon frequency of the propagating wave and the degree of perturbation. At low frequencies, the results match exactly with the finite element simulations, but at high frequencies, the 1-D model predictions deviate slightly from the finite element results.

2. PROBLEM DEFINITION

The problem considered in this paper is shown in Fig. 2, where sound from a line source propagates through a waveguide. Considering that the line source and the waveguide is long enough in the z direction, the pressure can be assumed to uniform in the z direction, and hence the analysis is restricted to the 2-D xy plane.

3. NUMERICAL FORMULATION

Sound propagation with cylindrical wavefront (3-D) is given by the Helmholtz equation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial z^2} + \left(\frac{\omega}{c}\right)^2 p = 0. \quad (3)$$

As explained in Fig. 2, the pressure variation in the z direction is not significant, and hence the equation reduces to Eq. (4) for the 2-D geometry:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \left(\frac{\omega}{c}\right)^2 p = 0. \quad (4)$$

The problem (Fig. 2) is symmetric, about the y-axis, therefore only the top half of the geometry is considered and is shown in Fig. 3. The waveguide considered is made of sound hard material with respect to air. The top surface of the waveguide is represented by the polar coordinates $(r,\theta(r))$ as shown in Fig. 3.

Sound propagation in such 2-D geometry is given by Eq. (4) which is a partial differential equation in pressure with respect to radial and angular coordinates. The aim in this work is to reduce the partial differential equation to an ordinary differential equation in the radial coordinate. For this, the numerical method of weighted residual method [13] is used, which is an integral method to obtain an approximate solution to any differential equation.

The weighted residual method can find an approximate solution to any differential equation by equating the integral of the equation with respect to a weighting function over a domain to zero. In the present case with Eq. (4), we chose to integrate the equation over a circular arc with center at the origin O and with an integrating variable ϕ. The variable angle ϕ varies across the arc from zero to $\theta(r)$. It will be shown later that the circular arc corresponds to the wavefront of the propagating wave with source placed at origin.

The implementation of weighted residual method on Eq. (4) is shown in Eq. (5) which uses weighting function as unity:

$$\int_0^\theta \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \left(\frac{\omega}{c}\right)^2 p\right) d\phi = 0. \tag{5}$$

The integral equation (Eq. (5)) can be split into three integrals for the ease of analysis:

$$I_1 = \int_0^\theta \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)\right) d\phi, \tag{6}$$

$$I_2 = \int_0^\theta \left(\frac{1}{r^2} r \frac{\partial^2 p}{\partial \phi^2}\right) d\phi, \tag{7}$$

$$I_3 = \int_0^\theta \left(\left(\frac{\omega}{c}\right)^2 p\right) d\phi. \tag{8}$$

As the waveguide is symmetric about the axis $\phi = 0$, the pressure is also symmetric and hence

$$\left.\frac{\partial p}{\partial \phi}\right|_{\phi = 0} = 0. \tag{9}$$

Using the Leibniz rule [14] and symmetric condition, the integrals can be simplified as

$$I_1 = \frac{1}{r}\left[\frac{d}{dr}\left[r\left\{\frac{d}{dr}\int_0^\theta p d\phi - p|_\theta \theta'\right\}\right] - \left.\left(r\frac{\partial p}{\partial r}\right)\right|_\theta \theta'\right], \tag{10}$$

$$I_2 = \left.\frac{1}{r^2}\frac{\partial p}{\partial \phi}\right|_\theta, \tag{11}$$

$$I_3 = \left(\frac{\omega}{p}\right)^2 \int_0^\theta p d\phi, \tag{12}$$

where prime denotes derivative with respect to r.

The upper surface (curve) of the waveguide is sound hard boundary condition which is given as

$$\hat{n}.\nabla p = 0, \tag{13}$$

where $\hat{n}$ is the surface normal at a point A as shown in Fig. 3. The expression when simplified leads to the following equation:

$$\left.\left[\frac{\partial p}{\partial r}\theta' - \frac{\partial p}{\partial \phi}\frac{1}{r^2}\right]\right|_\theta = 0. \tag{14}$$

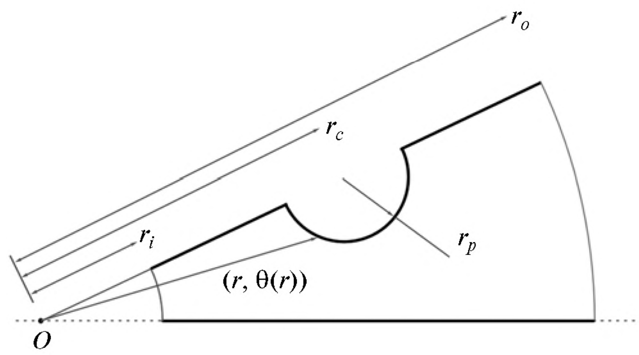

Fig. 4. Specific example of waveguide with perturbation of a semicircle.

As it can be seen by adding the three integrals, the sound hard boundary condition at the top surface appears in the expression leading to the following equation:

$$\frac{1}{r}\frac{d}{dr}\left[r\left(\frac{d}{dr}\int_0^\theta p d\phi - p|_\theta \theta'\right)\right] + \left(\frac{\omega}{c}\right)^2 \int_0^\theta p d\phi = 0. \tag{15}$$

The sound source is placed at the origin O, and assuming that the wave propagates with a circular wavefront, the pressure will be constant over any circular arc. With this assumption, pressure is now only a function of the radial distance. Therefore the above integral expression can be simplified to an ordinary differential equation in radial coordinate:

$$\frac{d^2 p}{dr^2} + \left(\frac{\theta'}{\theta} + \frac{1}{r}\right)\frac{dp}{dr} + \left(\frac{\omega}{c}\right)^2 p = 0. \tag{16}$$

The above equation is similar to the Webster horn equation (Eq. (2)), however, the above equation is valid for sound propagating with circular wavefront in a two dimensional waveguide.

4. VALIDATION OF THE 1-D MODEL WITH THE FINITE ELEMENT SIMULATION

Sound propagation through the 2-D waveguide can be modeled by the 1-D ordinary differential equation derived above. The purpose of this section is to validate the 1-D model by the 2-D finite element simulation and to observe the conditions for which the 1-D model predictions are comparable with the 2-D simulations.

To accomplish this goal, a specific example of a waveguide in the form of a sector is considered. A perturbation of a semicircle is introduced in the sector so that it perturbs the pressure field and forces it to be two dimensional. The symmetric part of the waveguide is shown in Fig. 4. A point source is located at the center O, which is also the center of the sector. The waveguide extends over a radius varying from r_i (0.1 m) to r_o (0.45 m). The perturbation of a semicircle is

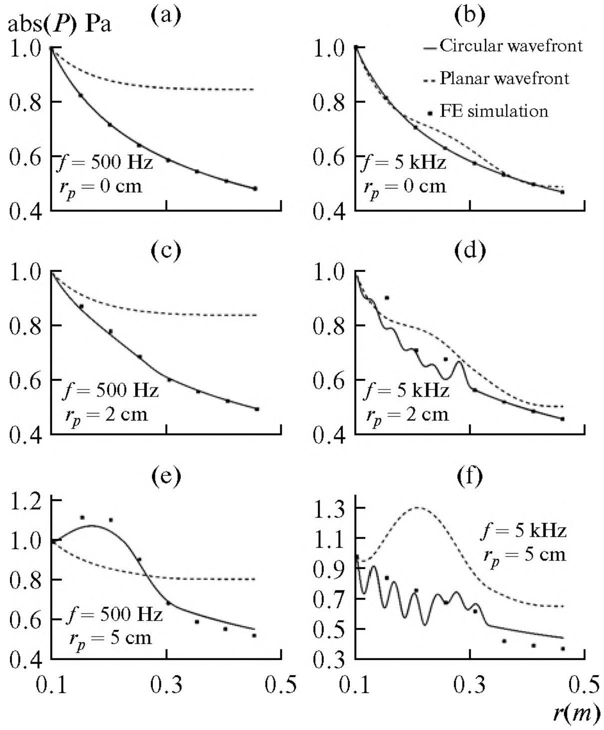

Fig. 5. Average pressure versus radial distance for wave propagating from a point source in a waveguide with circular wavefront, planar wavefront and finite element (FE) simulations.

introduced in the waveguide at a radial distance of r_c (0.28 m). This perturbation makes the waveguide non-uniform in the angular direction. Different waveguides were considered by varying the radius of semicircle r_p as zero, 2 and 5 cm.

For the same geometries of the waveguide, 2-D models were constructed in finite element software COMSOL Multiphysics. The solution to the 1-D differential equation was obtained by finite difference method by using second order central difference discretization. A pressure boundary condition of one Pascal was applied at the inlet (r_i), while radiation boundary condition was applied at the outlet (r_o) for both finite element and the 1-D model. For the 1-D model, radiation boundary condition was implemented by the 1-D Sommerfeld radiation boundary condition [15]

$$\frac{dp}{dr} - i\frac{\omega}{c}p = 0. \tag{17}$$

The average pressure $p(r)$ averaged over the angular coordinate ϕ from the 1-D model is plotted along with the finite element results for different geometries at 500 Hz and 5 kHz in Fig. 5. A comparison is also made with the Webster horn equation for the same geometries and frequencies.

The results (Fig. 5a and 5b) show that 1-D model predicts the average pressure exactly when the waveguide is uniform ($r_p = 0$). For this case, when there is no perturbation, the pressure is uniform in the angular direction and hence our assumption for the 1-D model holds exactly true. It can also be seen from the Eq. 16, when the term $\theta' = 0$, the equation reduces to the Bessel equation of zero order. And hence for this case, the 1-D model turns out to be an exact equation.

For the subsequent cases when the perturbation parameter, r_p is non zero, the waveguide is non-uniform in the angular direction, and so the pressure. Still the prediction by the 1-D model matches well with 2-D finite element model. At low frequency (500 Hz) or when the perturbation (r_p) is small, the pressure from 1-D model is quite close to the 2-D finite element results (Fig. 5a–5e). However, the results vary slightly, especially for highly non-uniform waveguide at high frequency (Fig. 5f). For such a case, we need to include higher modes of pressure in the angular direction to get a better solution.

5. CONCLUSION

A one dimensional model for sound propagation from a point source through a two dimensional waveguide has been proposed. The model assumes uniform pressure across the circular wavefront and is obtained from integrating the governing Helmholtz equation in polar coordinates using weighted residual method. The results obtained are compared with 2-D finite element simulation for different waveguide geometries and frequencies. The results are in good agreement. However, at high frequencies and for non-uniform waveguide, the one dimensional model prediction differs slightly from the finite element results. The 1-D model can be improved in the future by including a non-uniform pressure in the angular coordinate to obtain higher order variants of this 1-D model.

It also interesting to note that the same equation can be obtained from the general form of the Webster horn equation derived in the Pierce's textbook [16]. The general form of Webster equation, using the present convention of variables is

$$\frac{1}{S(x)}\frac{\partial}{\partial x}\left(S(x)\frac{\partial p}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}. \tag{18}$$

Considering the area $S(x)$, as a function of r, which can be written as $S(r) = r\theta(r)$. Substituting this in the general equation and for a harmonic analysis, the equation after some manipulation leads to the Eq. (16) derived in this paper.

The equation for wave propagation with circular wavefront provides with a simple yet accurate model for sound propagating from the line source, enclosed by some sound hard surface acting as a waveguide. The predictions by this 1-D model are accurate at low frequency and for less distorted waveguide.

REFERENCES

1. E. Lawrence and A. R. F. Kinsler, *Fundamental of Acoustics*, 3rd ed., (Wiley, California, 1982).
2. A. G. Webster, Proc Nat. Acad. Sci. USA, **5**, 275 (1919).
3. E. Eisner, J. Acoust. Soc. Am. **41**, 1126 (1967).
4. P. A. Martin, J. Acoust. Soc. Am. **116**, 1381 (2004).
5. S. W. Rienstra, Siam J. Appl. Mathem. **65**, 1981 (2005).
6. A. Gupta, K.M. Lim, and C.H. Chew, Appl. Phys. Lett. **98**, 20 (2011).
7. A. Gupta, K. Lim, and C. Chew, J. Acoust. Soc. Am. **132**, 2909 (2012).
8. A. Gupta, K. Lim, and C. Chew, Appl. Mech. Mater. **152**, 281 (2012).
9. A. Gupta, C. Chew, and K. Lim. *Proc. 4th Int. Conf. on Experimental Mechanics, 2009*, Int. Soc. Optics Photonics.
10. V. F. Kovalev and O. V. Rudenko, Acoust. Phys. **58**, 269 (2012).
11. V. Khitrik, Acoust. Phys. **46**, 613 (2000).
12. I. Makarov, Acoust. Phys. **55**, 261 (2009).
13. O. C. Zienkiewicz, in *The Finite Element Method Set*, 6th ed. (Butterworth-Heinemann, Oxford, 2005) pp. 54–102.
14. H. Flanders, The Am. Mathem. Monthly **80**, 615 (1973).
15. D. Givoli, J. Comp. Phys. **94**, 1 (1991).
16. A. D. Pierce, *Acoustics: An Introduction to Its Physical Principles and Applications. McGraw-Hill Series in Mechanical Engineering* (McGraw-Hill, New York, 1981).