ISSN 1063-7710, Acoustical Physics, 2011, Vol. 57, No. 4, pp. 465–472. © Pleiades Publishing, Ltd., 2011.

PHYSICAL ACOUSTICS

Can't One Really Hear the Shape of a Drum?[1]

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Received September 7, 2010

Abstract—In this paper we study the wave and the Klein–Gordon equations in frontiers with the same set of eigen values using a computational algorithm based on the finite difference method and the discrete Fourier transform. Doing this we found that although the set of eigen values in the two shapes are equal, the intensities in the spectrum are different, which means that the question, can one hear the shape of a drum? is still *open*.

Keywords: eigen states, eigen values, Fourier transform.

DOI: 10.1134/S106377101104021X

INTRODUCTION

When you hear a band you are able to distinguish each instrument by its sound. In drums, the sound is determined by the material, how tight this is, and by its shape. A very interesting question that arises then is to determine how much information can be obtained of the sound produced by the drum. This problem was discussed by Mark Kac, who in 1966 published a paper called Can one hear the shape of a drum? [1, 2, 8] where he explores the possibility of acknowledging the shape of a drum just by knowing the set of normal vibrating frequencies. During the same epoch this problem was solved by John Milnor, who shows two regions in dimension 16 that share the same set of eigen values for the operator $-\nabla^2$. In one dimension it is obvious that the set of eigen values is plenty enough to know the length of "the drum". But in two dimensions the problem remained without solution till 1992, when Gordon and Web showed in their paper "You can't hear the shape of a drum" [3] two regions that share the same set of eigen values for the operator $-\nabla^2$.

Among the things that can be acknowledged by the sound of a drum, there are two meaningful results, one says that it is possible to know the drum's area. This was a conjecture proposed by Lorentz and proved by Herman Weyl. He was capable to prove it using the theory of integral equation that his teacher Hilbert had developed a few years before. The equation is as follows:

$$\lim_{\lambda \to 0} \frac{N(\lambda)}{\lambda} = \frac{|\Omega|}{2\pi}, \tag{1}$$

where $|\Omega|$ is the area of the drum and $N(\lambda)$ is the number of eigen values less than λ. And the second is the possibility to determine which kinds of conditions are imposed on the frontier, as can be seen in [4, 21].

PRELIMINARIES

The Klein–Gordon equation (K–G) is the relativistic version of the Schrödinger equation for a particle without spin. It was proposed by Oskar Klein and Walter Gordon and can be obtained taking the Hamiltonian relativistic of a free particle:

$$H = \sqrt{p^2c^2 + m^2c^4}. \tag{2}$$

Rewriting this equation in the terms of the operators, we can get to

$$i\hbar\frac{\partial\Psi}{\partial t} = \sqrt{-\hbar^2c^2\nabla^2 + m^2c^4}\Psi. \tag{3}$$

But the square root in this equation has some problems, like the relativistic invariance is not clear due to the lack of symmetry between space and time coordinates. To solve this problem and get rid of the square root, it is common to use the Hamiltonian square like this:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\Psi = 0. \tag{4}$$

This is known as the Klein–Gordon equation, and as we will show the base states are the same of the operator $-\nabla^2$, so the results of Gordon and Webb, see [3], where they proved that it is impossible to know the shape of a drum just by knowing its set of eigen values, is also valid for the K–G equation. But knowing the set of eigen values is not enough to determine how is going to be the sound produced by the drum; this information only gives us a hint of how the frequencies that are going to compose the sound of the drum are. To have a

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real knowledge of the drum's sound, we must have information about how intense are these eigen frequencies in the sound perceived by our ears. In order to do this, we are going to study the temporal evolution of the wave and the Klein–Gordon equations.

NUMERICAL METHOD

Here we solved the wave and the Klein–Gordon equations finding the acceleration of each point on the grid trough finite difference method as follows: first we started by discretizing the spatial derivate transforming the equations in

$$\frac{\partial^2\Psi}{\partial t^2} = f(\Psi^i_{i,j}, \Psi^i_{i-1,j}, \Psi^i_{i+1,j}, \Psi^i_{i,j-1}\Psi^i_{i,j+j}). \quad (5)$$

Where the sub-indices denote space and the super-indices denote time. Therefore we have an expression for the acceleration for each point on the grid and using the Euler method we calculate the position of each cell for each time step.

Doing this we are able to set the function $\Psi(x, y, t)$ as a vector of N^2 components. Of course we are thinking in a two dimensional case where the square drum has been represented by a grid of $N \times N$. Therefore the function $\Psi(x, y, t)$ is written as $\Psi^t_{i,j}$ and doing this the wave and the Klein–Gordon equations are transformed in:

$$\vec{\Psi}^{i+1} = A\vec{\Psi}^i + B\vec{\Psi}^{i-1}, \quad (6)$$

where A and B are operators that operate over the estates $\vec{\Psi}^i$. Now, it is very important to notice that it is possible not only to advance in time, but to go backwards in time, writing $\vec{\Psi}^{i-1}$ in terms of $\vec{\Psi}^{i+1}$ and $\vec{\Psi}^i$. More about similar vibrational problems [19, 20] and this kind of methods and its convergence can be seen, for example, in [5, 6, 10, 11].

METHOD FOR FINDING THE EIGEN VALUES AND EIGEN FUNCTIONS FOR THE KG AND THE WAVE EQUATIONS

Now we want to know the set of the eigen functions and the eigen values. In the K–G equation, we are trying to find the energies available of a particle confined in a two dimensional box and its quantum states, and in the case of the wave equation we are looking for the normal modes of oscillation and its frequencies.

So the general problem will be to solve the following equation:

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \frac{1}{c}\frac{\partial^2\Psi}{\partial t^2} - \frac{m^2c^2}{\hbar^2} = 0 \quad (7)$$

on an region under Dirichlet conditions in the frontier for a certain initial condition $f(x, y)$.

But before we begin to explain the way to find its eigen values and eigen functions, it is important to notice that for the K–G equation, the function $\Psi(x, y, t)$ could be written as $R(x, y)T(t)$ transforming the K–G equation in two ordinaries differential equations like

$$\frac{dT^2}{dh^2} = -w_n^2 T, \quad \nabla^2 R = -\frac{w_n^2 - \alpha}{c^2}R, \quad (8)$$

where $\alpha = m^2c^4/\hbar^2$.

So the states available for the particle are the same as the normal states of a classical drum, with the difference that the oscillation frequencies can change depending on the constant. More about the K–G equation can be found in [9, 13–16].

Now we are going to find the eigen functions R and its eigen values. In order to do this, we solved the problem proposed in (7) using our algorithm. Remember that in the case of the wave equation $m = 0$.

But, of course, finding the solution $\Psi(x, y, t)$ for any time it is not enough to get to our main target, but this is the path to get to it. We are going to have into account the fact that any state of the system can be written as a linear combination of the basis state, so we are going to put the system in an initial state $f(x, y)$ and then evolve it for a long time, so we will calculate the Fourier transform of one point of the system. Therefore we obtain set of frequencies that compose the whole oscillation, and in the case of the K–G equation the set of energies available for the particle enclosed.

In our algorithm it is important to be careful with the point on which we choose to calculate the Fourier transform, because if we choose a point in which the eigen function that we are interested in has a node, we are not detecting its eigen value. Once we have the eigen values we are going to reproduce the eigen functions. In order to do that, we will take advantage from the system's resonance, which means that forcing one point with its natural frequencies (eigen values) we obtain the eigen functions.

To close this section, it is important to say that the forced point has to be very well chosen. It must not be a spot where we expect to find a node because in other case we will not get the real shape of the eigen function, and neither a point where a maximum is expected, because doing this we are forcing this particular point to have the same maximum for any time, and also we would be changing the shape of the frontier. Due to this the forced point must belong to the frontier or it must be as close to it as possible.

COMPARISON WITH A SIMPLE ONE DIMENSIONAL CASE FOR THE K–G EQUATION

For this case we will work with natural units, so the K–G equation in one dimension will be:

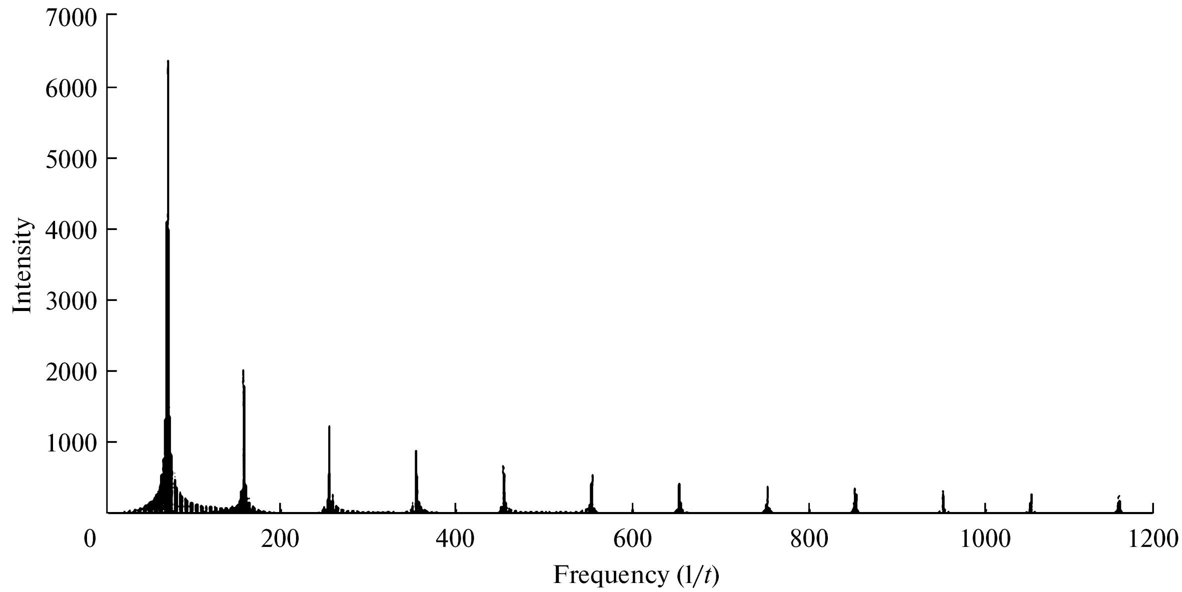

Fig. 1. Frequency spectrum for a particle enclosed in a one dimensional box.

$$\nabla^2\Psi - \frac{\partial^2\Psi}{\partial t^2} - m^2\Psi = 0. \quad (9)$$

In this case, the constant m^2 will be 10^5 and the particle will be enclosed in the interval (0, 0.01). In other words, we have a particle of $\sqrt{10^5}$ MeV enclosed in one dimensional box of length 0.01 MeV^{-1}.

In order to find the eigen values w_n, we analyze the point $x = 0.005$, which means that we are not going to detect the energies that belong to the even normal states. The spectrum given by our algorithm is shown on Fig. 1.

In order to show how accurate our method is, it is important to obtain the frequencies w_n in an analytical way. So the solutions for the Eq. (8) are sin and cos functions and due of the Dirichlet boundary condition we have:

$$\sqrt{\frac{w_n^2 - \alpha}{c^2}} = \frac{n\pi}{L}, \quad (10)$$

where L is the length of the box, so the angular frequencies w_n are:

$$w_n = \frac{\sqrt{\alpha L^2 + n^2\pi^2 c^2}}{L}. \quad (11)$$

Now, the following table shows some of the frequencies given by the formula (11) and the frequencies given by our algorithm.

As we can see in the Table 1, the method used to find the eigen values and, in consequence, the energies available for the particle it is very accurate. Therefore we are going to use it in the two dimensional cases for a complex boundary.

DRUMS WITH THE SAME SET OF EIGEN VALUES

Inspired in the work done by Gordon and Webb in their paper [3], we took the same shapes which are shown on Fig. 2. It is very important to say that in this case we are not using any kind of units, we are just solving the Eq. (7), or the Eq. (8), where $c = 10$, $\alpha = 10^5$, $a = 0.025$.

Using our algorithm we calculated the frequencies (energies) for a particle enclosed in each region. These results are shown in the Fig. 3, but due to the difficulty to conclude if the spectrums are equal or not, in the Table 2 we show the frequencies for the first 4 peaks.

First, is important to say that in this moment we are not interested in the intensities of the spectra, but in which frequencies the peaks are. Second, to note that the two regions shown in the Fig. 2 shares the same set

Table 1. Comparison between our algorithm and the analytical results

Quantum number n	Frequency (analytical), Hz	Software frequency, Hz	Error percentage, %
1	70.94	71	0.084
3	158.21	158	0.131
5	255.01	255	0.009
7	353.6	354	0.113

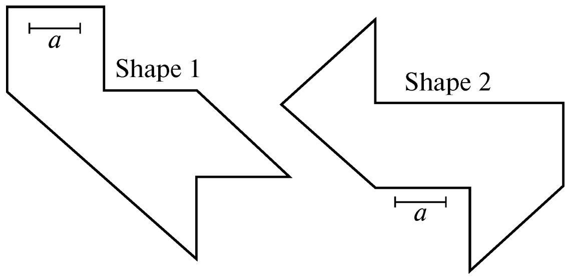

Fig. 2. Boundaries with the same set of eigen values.

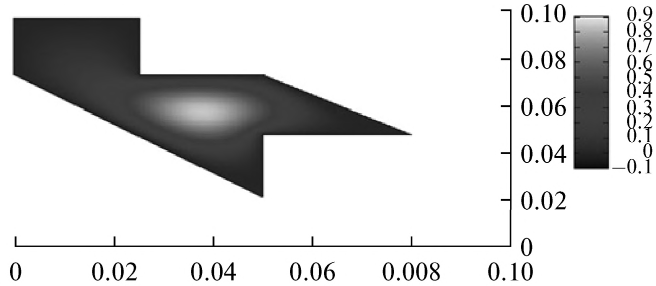

Fig. 4. First base state for the shape 1.

of eigen values, due to the fact that in [3] Gordon and Webb proved that this two shapes shares the same set of eigen values for the operator $-\nabla^2$. It is clear that the frequencies shown in the Table 2 are actually eigen values, because forcing the system whit this frequencies we obtain the same well know eigen functions for this drums, as can be seen in the Figs. 4–7 and in the paper [7]. So in the Fig. 3 and Table 2 we are corroborating their results for the K–G equation, which is very similar to the wave equation once the variables separation is made, see Eq. (8).

This means that is impossible to notice the difference between the two shapes just by knowing the set of energies available for the particle. But as can be seen in the Fig. 3, the intensities in the spectra for each shape are very different. This gives us a hint that it is maybe possible to notice the difference of the shapes not only by knowing the set of eigen values, but its intensities. Now, it is clear that the energies available for the particle enclosed in the two shapes are equal, but going back to the classical drum studied by Gordon and Webb, what does it mean to hear a drum? Well, it is not enough to know each one of the fundamental frequencies composing the whole oscillation. Also you have to know the intensity of each frequency. Mathematically speaking any kind of oscillation can be written as

$$\Psi(x, y, t) = \sum_{i=1}^{\infty} C_i T_i(w_i, t) R_i(x, y), \qquad (12)$$

where C_i are the intensities; $T_i(w_i, t)$ are the functions responsible for the oscillation with frequencies w_i, and finally $R_i(x, y)$ are the eigen function of the operator $-\nabla^2$. These coefficients C_i must be equals for the two drums in order to make both drums sound alike. Therefore we are going to concentrate ours efforts in finding C_i to compare these coefficients in the two drums for a particular oscillation. To achieve this goal we are going to reproduce the base states for the two drums in order to find where they have its maximum.

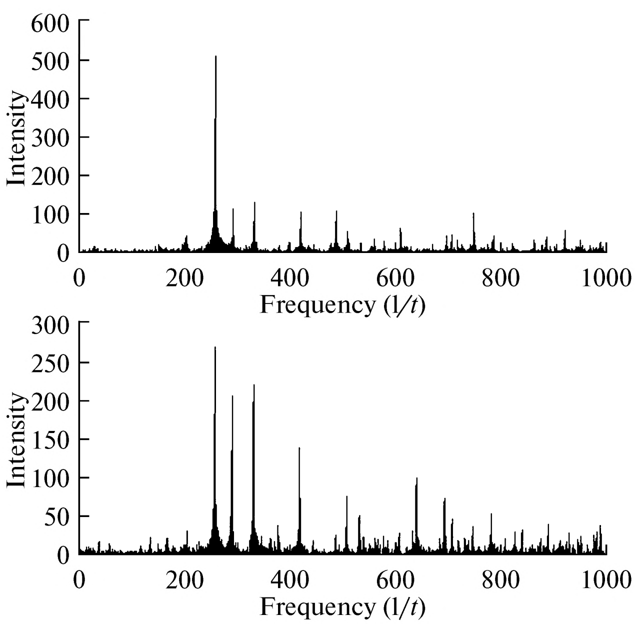

Fig. 3. Frequency spectrums for the shapes 1 and 2 in the Fig. 2.

BASE STATES FOR THE QUANTUM DRUMS

Using the resonance of the system, we forced it with the first and second resonance frequencies for the two shapes, in a spot near the frontier, achieving the base states in both cases as is shown in the Figs. 4–7.

As we expected, the base states are the same as the ones for the classical wave equation, see [7], and that more important for us now, we know where the maximum for each eigen function is. But before we go on

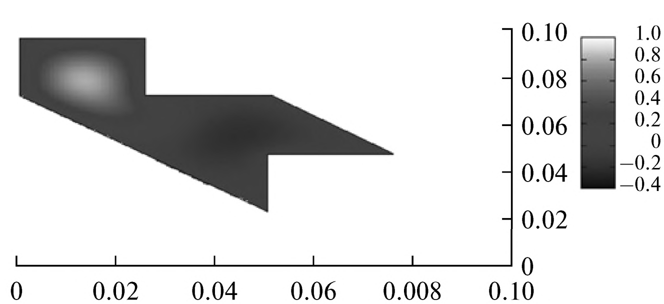

Fig. 5. Second base state for the shape 1.

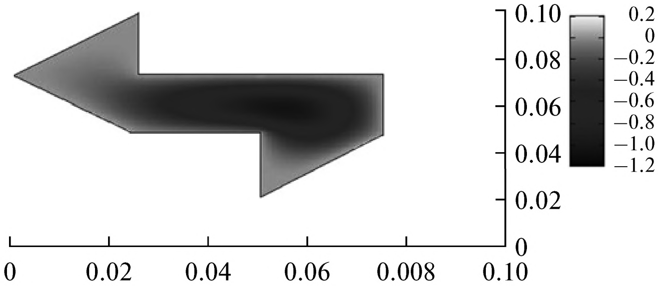

Fig. 6. First base state for the shape 2.

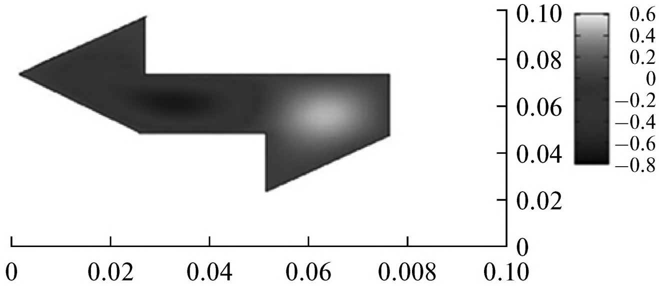

Fig. 7. Second base state for the shape 2.

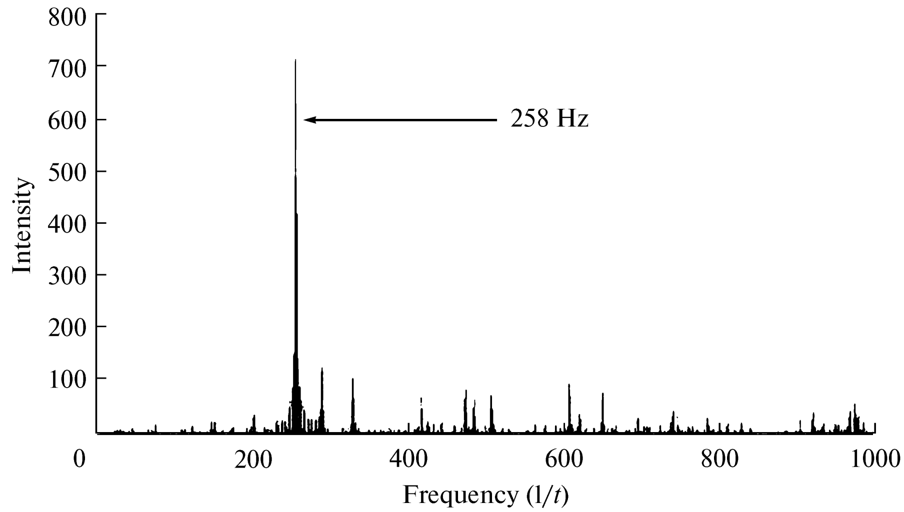

Fig. 8. Frequency spectrum for the shape 1, taken where the first state achieves its maximum in the K–G equation.

with the discussion it is very important to say that we know that the Figs. 4 and 6 are the firsts base states because it is well known that for the operator $-\nabla^2$ for Dirichlet boundary condition and with a smooth frontier, the first eigen function do not change its sign on the region, this can be seen in [17, 18].

SPECTRUM INTENSITIES

Now that we know where the maximums are, we are going to calculate the Fourier transform in this points, taking as the initial state of the system the function $f(x, y) = 1$ in order to know the coefficients C_i, which finally will tell us if the two drums sound alike. It is important to note that the initial condition $f(x, y) = 1$ is arbitrary. We can choice a different initial condition, but this must be the same for both shapes.

The Figs. 8 and 9 show the spectrums taken where the maximum of the firsts eigen states are for the shapes 1 and 2. As can be seen, the intensities of the first peak, frequency 258 Hz, are very similar, which means that the coefficients C_1, Eq. (12), are almost equals in both quantum drums, and therefore we can't decide if they "sound" alike or not. But doing the same procedure for the second base state we found a very different intensities in the second peak, 292 Hz, Figs. 10 and 11, which means that the behavior of the oscillation for both drums is very different.

Inspired on these results, we did an analogous analysis for the wave equation, in this case the velocity of propagation was 10 m/s, realizing that for the seconds eigen states, as in the quantum case, the peaks in both spectra, peaks of 244 Hz, are very different in their

Table 2. Comparison between the frequency spectra for the shapes 1 and 2

	Shape 1	Shape 2
Frequencies for the peak 1, Hz	258	258
Frequencies for the peak 2, Hz	292	291
Frequencies for the peak 3, Hz	331	331
Frequencies for the peak 4, Hz	419	418

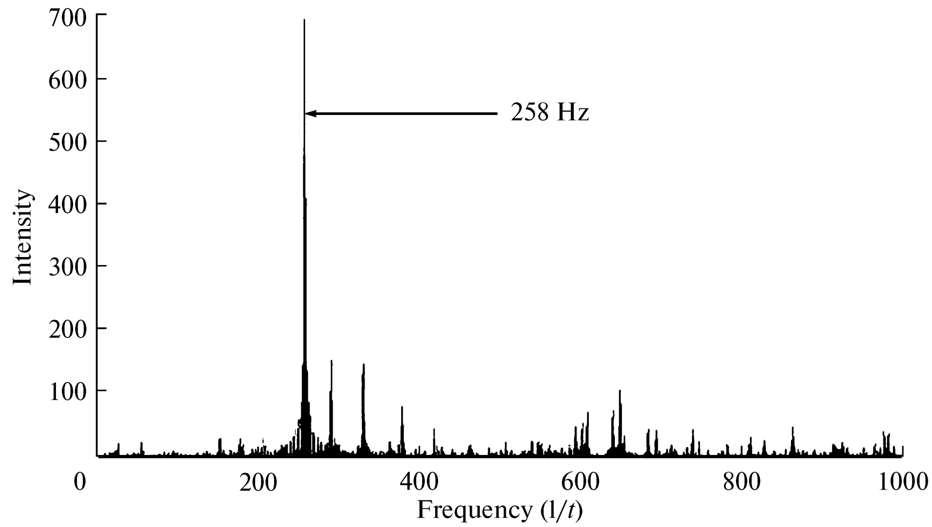

Fig. 9. Drequency spectrum for the shape 2, taken where the first state achieves its maximum in the K–G equation.

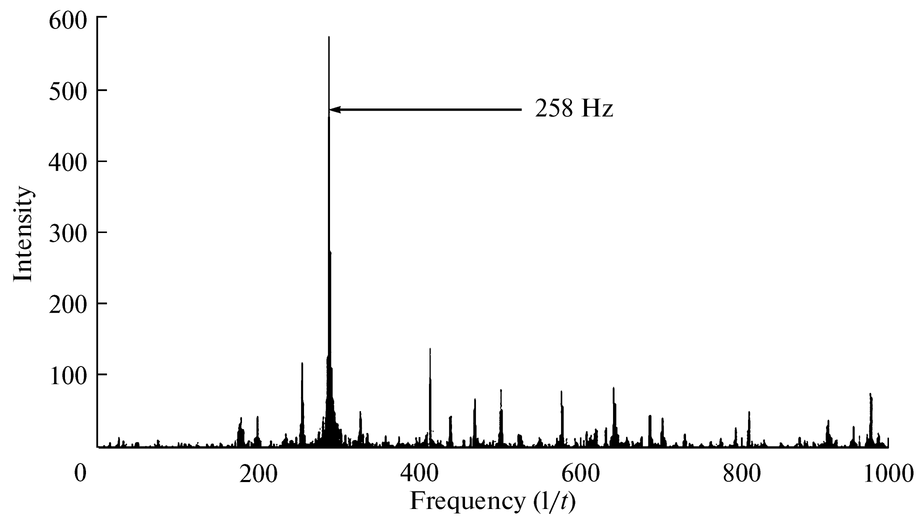

Fig. 10. Frequency spectrum for the shape 1, taken where the second state achieves its maximum in the K–G equation.

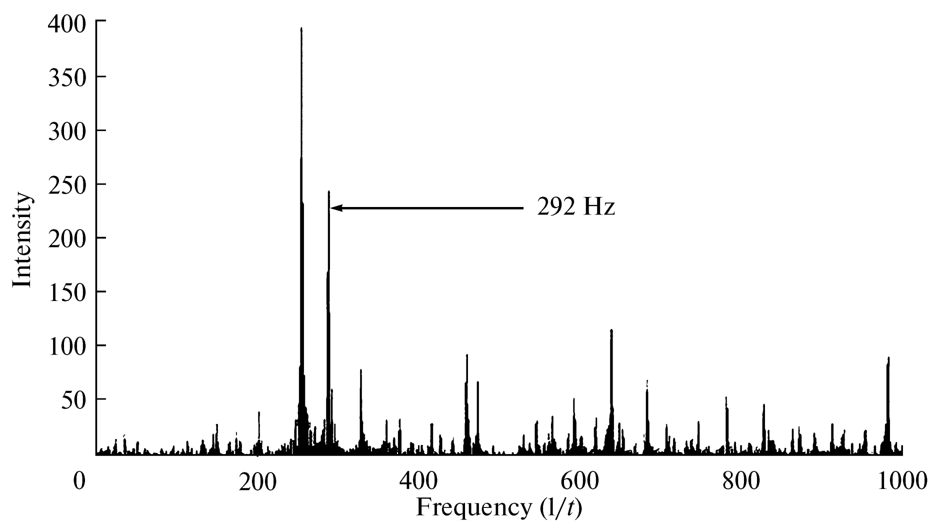

Fig. 11. Frequency spectrum for the shape 2, taken where the second state achieves its maximum in the K–G equation.

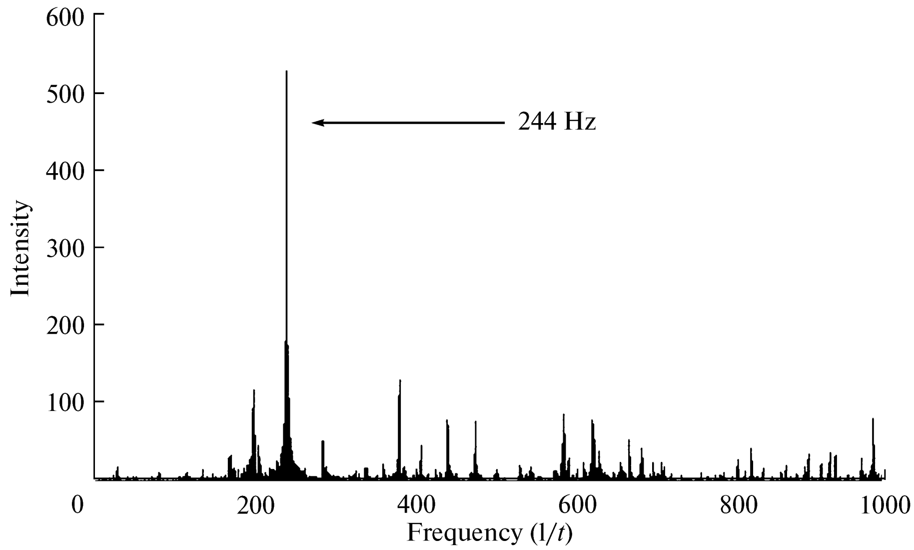

Fig. 12. Frequency spectrum for the shape 1, taken where the second sates achieves its maximum for the classical wave equation.

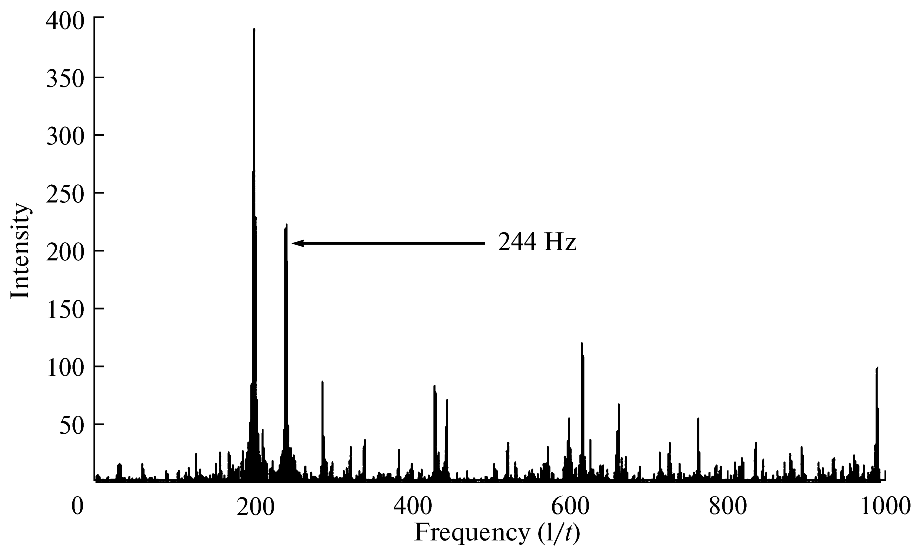

Fig. 13. Frequency spectrum for the shape 2, taken where the second sates achieves its maximum for the classical wave equation.

intensities, Figs. 12 and 13, which means that there is a big difference in the coefficients C_2, Eq. (12), for the two drums. As a consequence, we can differentiate both drums by their sound, due to the fact that the sound produced by the first drum is going to be sharper.

CONCLUSIONS

We studied the behavior of the Klein Gordon and classical wave equation and far from solving a problem we found another one, in which two regions that were sharing the same set of eigen values for the operator $-\nabla^2$ sound different. This means that both drums can be distinguished by the sound they produce. Therefore it let us understand that the question: "Can one hear the shape of a drum?" has not been answered.

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