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Dispersion of Elastic Waves in Microinhomogeneous Media and Structures

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Abstract—The propagation of elastic waves in periodic stratified media with arbitrary local anisotropy and in anisotropic plates and bars inhomogeneous in thickness is considered under the condition that the ratio of the scale characterizing the inhomogeneity of the medium or the thickness of a plate or bar in thickness to the typical wavelength is small. The propagation of long waves is described using the effective averaged equations with high-order accuracy, which are derived by the method of two-scale asymptotic expansions in ε. The results of analytic and numerical studies of their principal terms responsible for the dispersion of waves are presented. The form of the dependence of the wave velocity on the wavelength is studied for structures with different types of symmetry.

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INTRODUCTION

The processes that occur in inhomogeneous media when the scale of inhomogeneity is much smaller than the global scale of the problem are studied using models that appear as a result of a certain averaging. For a medium with a periodic structure, one of the methods of obtaining effective averaged equations that require no preliminary hypotheses concerning possible types of the local stress–strained state is the method of two-scale asymptotic expansions, which was developed in the mathematical theory of averaging [1, 2]. This algorithm allows one not only to construct the equations for the period average quantities, but also to determine the local fields to a certain approximation. It should be noted that the global properties of a microinhomogeneous medium may qualitatively differ from the properties of its constituents [3].

The present paper is devoted to derivation and investigation of equations describing the propagation of elastic waves in periodically stratified—in particular, layered—media and also in plates and bars inhomogeneous in thickness under the condition that the ratio ε of the scale of inhomogeneity occurring in a medium or a plate or bar to the typical wavelength is small: $\varepsilon \ll 1$.

The theory of fine-layered media is important for acoustics and seismology. It has been studied by many authors [4–9]. The main purpose of these studies was to determine the effective elastic properties of such media. Publications devoted to models with dispersion [10, 11] only considered media that consisted of a repeated set of two homogeneous isotropic layers. The present paper studies stratified (not necessarily layered) media with an arbitrary local anisotropy. The main purpose of this paper is derivation and investigation of equations with dispersion. The equations are derived using the method of two-scale asymptotic expansions.

If, in the method of two-scale asymptotic expansions, the derivation of the equations is restricted to a zero approximation in ε, then, for linearly elastic media, after averaging, one obtains the equations corresponding to ordinary (anisotropic in the general case) elastic media with some effective elastic moduli. To describe the dispersion of waves, it is necessary to take into account the terms of higher orders in ε. In this case, the equations contain higher-order derivatives of displacements with respect to coordinates and time [1]. Another effect that requires equations with higher accuracy in ε is the so-called scale effect: the effective properties of a medium depend on the size of the inhomogeneities even when this size is much smaller than the size of the body. Equations with high accuracy in ε are necessary for describing the processes in narrow zones, e.g., the structure of a shock wave [12].

The models of compressible liquids, the equations of which contain higher-order derivatives of various parameters of the medium with respect to time and coordinates, were introduced phenomenologically, in particular, in [13, 14]. Equations with higher-order derivatives are also used in the models of the Cosserat theory of elasticity [15, 16]. As a rule, these equations are postulated on the basis of phenomenological hypotheses. The method of two-scale asymptotic expansions allows one to derive the expressions for the coefficients of these equations in explicit form when the microstructure of the medium is known. After some modification, the method of two-scale asymptotic expansions can be used to derive the averaged equations of plates and bars if the wavelengths of the waves under study are much greater than the typical thickness of the plate or bar.

The present paper briefly describes the method used to derive the averaged equations and presents the results of analytic and numerical studies of their principal terms, which determine the dispersion of waves in various inhomogeneous media and structures. The main part of the paper is a brief review of the results obtained together with N.S. Bakhvalov that have been reported in a number of recent publications [17–23]. The numerical study of the properties of the derived equations was carried out with participation of K.Yu. Bogachev and A.E. Yakubenko.

METHOD OF TWO-SCALE ASYMPTOTIC EXPANSIONS FOR DERIVATION OF AVERAGED EQUATIONS OF MICROINHOMOGENEOUS ELASTIC MEDIA

Let us write the equations of the linear theory of elasticity in the form

$$L\mathbf{u} = -\rho\frac{\partial^2\mathbf{u}}{\partial t^2} + \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial\mathbf{u}}{\partial x_j}\right) = 0.$$

Here, $\mathbf{u}$ is the displacement vector and A_{ij} are the matrices of elastic coefficients. It is assumed that they, as well as the density ρ, are periodic functions of coordinates with a period d. We consider the waves the typical wavelength l of which is much greater than d, so that $\varepsilon = \frac{d}{l} \ll 1$. Let us introduce slow x_j and fast y_j dimensionless variables according to the formulas

$$x_j = \frac{\bar{x}_j}{l}, \quad y_j = \frac{\bar{x}_j}{d} = \frac{x_j}{\varepsilon},$$

where $\bar{x}_j$ are the dimensional coordinates. For the media under consideration, we have

$$\rho = \rho(y_j), \quad A_{ij} = A_{ij}(y_j).$$

The displacement vector is assumed to be a function of both fast and slow variables, $\mathbf{u} = \mathbf{u}(t, x_j, y_j, \varepsilon)$, and is representable in the form of an asymptotic series expansion in powers of ε. In [1], this series was shown to have the form

$$\mathbf{u} \sim \mathbf{v} + \sum_{m=1}^{\infty}\varepsilon^m N^q_{l_1l_2l_3}(y_j)\frac{\partial^m\mathbf{v}}{\partial t^q\partial x_1^{l_1}\partial x_2^{l_2}\partial x_3^{l_3}}, \tag{1}$$

where $m = q + l_1 + l_2 + l_3$; $N^q_{l_1l_2l_3}$ are periodic functions of fast variables, these functions being determined by the structure of the medium; and $\mathbf{v} = \mathbf{v}(x_1, x_2, x_3, t)$ is independent of the fast variables. The substitution of series (1) in the initial equation yields

$$L\mathbf{u} \sim \sum_{m=0}^{\infty}\varepsilon^{m-2}H^q_{l_1l_2l_3}(y_1, y_2, y_3)\frac{\partial^m\mathbf{v}}{\partial t^q\partial x_1^{l_1}\partial x_2^{l_2}\partial x_3^{l_3}}, \tag{2}$$

where

$$\begin{aligned} H^q_{l_1l_2l_3} &= \frac{\partial}{\partial y_i}\left(A_{ij}\frac{\partial N^q_{l_1l_2l_3}}{\partial y_j}\right) - \rho N^{q-2}_{l_1l_2l_3} \\ &+ A_{ij}\frac{\partial N^q_{l_1-\delta_{i1}, l_2-\delta_{i2}, l_3-\delta_{i3}}}{\partial y_j} \\ &+ \frac{\partial(A_{ij}N^q_{l_1-\delta_{j1}, l_2-\delta_{j2}, l_3-\delta_{j3}})}{\partial y_i} \\ &+ A_{ij}N^q_{l_1-\delta_{i1}-\delta_{j1}, l_2-\delta_{i2}-\delta_{j2}, l_3-\delta_{i3}-\delta_{j3}}, \end{aligned} \tag{3}$$

and δ_{ij} are the Kronecker deltas. It is possible to determine $N^q_{l_1l_2l_3}$ so that, in Eq. (2), the coefficients multiplying the negative powers of ε are zero while the coefficients multiplying its nonnegative powers are identical to certain constants:

$$\begin{aligned} H^q_{l_1l_2l_3}(y_1, y_2, y_3) &= 0 \text{ for } m = q + l_1 + l_2 + l_3 < 2, \\ H^q_{l_1l_2l_3}(y_1, y_2, y_3) &= h^q_{l_1l_2l_3} = \text{const for } m \geq 2. \end{aligned} \tag{4}$$

Here,

$$h^q_{l_1l_2l_3} = \langle H^q_{l_1l_2l_3}\rangle.$$

The angular brackets $\langle . \rangle$ denote the period average values. The following condition is usually set is usually set as an additional condition necessary for unique determination of $N^q_{l_1l_2l_3}$:

$$\langle N^q_{l_1l_2l_3}\rangle = 0.$$

Then, $\mathbf{v} \sim \langle\mathbf{u}\rangle$.

The averaged equation of an infinite order of accuracy in ε has the form

$$\bar{L}\mathbf{v} \sim \sum_{m\geq 2}\varepsilon^{m-2}h^q_{l_1l_2l_3}\frac{\partial^m\mathbf{v}}{\partial t^q\partial x_1^{l_1}\partial x_2^{l_2}\partial x_3^{l_3}} \sim 0,$$
$$m = q + l_1 + l_2 + l_3.$$

Equations (4) are the equations for determining $N^q_{l_1l_2l_3}$. From the structure of formulas (3), it follows that, for determining $N^q_{l_1l_2l_3}$, it is necessary to solve static problems of the theory of elasticity within the periodicity cell with some special body forces and boundary conditions. In the case of a complicated cell structure, these problems can only be solved numerically. It is important that these problems do not depend on the global initial and boundary conditions. For a given structure, they should be solved once, after which the coefficients of the averaged equations are calculated. Then, it is possible to solve specific problems on the behavior of the medium under any given conditions.

Let us list some of the properties of the averaged equations for arbitrary elastic periodic media [24].

(i) The averaged equations contain no odd derivatives with respect to time.

(ii) The matrices of the coefficients multiplying the even derivatives with respect to coordinates are symmetric, whereas the matrices of the coefficients multiplying the odd derivatives are antisymmetric:

$$(h^q_{l_1l_2l_3})^T = (-1)^{l_1+l_2+l_3} h^q_{l_1l_2l_3}.$$

(iii) It is possible to obtain different asymptotically equivalent forms of higher-order equations by using the equations of lower approximation. In particular, it is possible to eliminate the time derivatives of orders higher than second. In this case, it is possible either to retain the mixed derivatives with respect to t and x_i (equations in the form A) or to obtain equations containing no mixed derivatives with respect to t and x_i (the form B).

WAVES IN STRATIFIED MEDIA AND PLATES

Let us consider a stratified medium [17–20] for which

$$\rho = \rho(y), \quad A_{ij} = A_{ij}(y), \quad y = x_1/\varepsilon.$$

The equations for the waves propagating in the direction perpendicular to the layers are as follows: $\mathbf{u} = \mathbf{u}(t, x_1, y)$, $\mathbf{v} = \mathbf{v}(t, x_1)$. Let us use the notation

$$x_1 = x, \quad h^q_{l00} = h^q_l, \quad A_{11} = A,$$
$$\hat{\rho} = \langle \rho \rangle, \quad \hat{A} = \langle A^{-1} \rangle^{-1}.$$

Then, we have

$$\bar{L}\mathbf{v} \sim -\hat{\rho}\frac{\partial^2 \mathbf{v}}{\partial t^2} + \hat{A}\frac{\partial^2 \mathbf{v}}{\partial x^2} + \sum_{q+l=3}^{\infty} \varepsilon^{q+l-2} h^q_l \frac{\partial^{q+l}\mathbf{v}}{\partial t^q \partial x^l} \sim 0.$$

It can be shown [1, 18] that $h^0_n = 0$ for $n > 2$. Therefore, rejecting the terms the order in ε of which is higher than second, we obtain

$$\hat{L}\mathbf{v} = -\hat{\rho}\frac{\partial^2 \mathbf{v}}{\partial t^2} + \hat{A}\frac{\partial^2 \mathbf{v}}{\partial x^2} + \varepsilon h^2_1 \frac{\partial^3 \mathbf{v}}{\partial t^2 \partial x} + \varepsilon^2\left(h^4_0 \frac{\partial^4 \mathbf{v}}{\partial t^4} + h^2_2 \frac{\partial^4 \mathbf{v}}{\partial t^2 \partial x^2}\right) = 0. \tag{5}$$

In addition, we write the equation in the form A:

$$L^A\mathbf{v} = -\hat{\rho}\frac{\partial^2 \mathbf{v}}{\partial t^2} + \hat{A}\frac{\partial^2 \mathbf{v}}{\partial x^2} + \varepsilon h^2_1 \frac{\partial^3 \mathbf{v}}{\partial t^2 \partial x} + \varepsilon^2 \tilde{h}^2_2 \frac{\partial^4 \mathbf{v}}{\partial t^2 \partial x^2} = 0. \tag{6}$$

For the type of layered medium under study, in [18] we determined the functions $N^q_{l_1l_2l_3}$ in explicit form and the formulas for the coefficients multiplying the higher derivatives in Eqs. (5) and (6), which contained the integrals of certain functions of ρ and A over the period. Analysis of these formulas suggests the following conclusions.

(i) When A is a scalar (e.g., waves in a layered liquid), we have $h^4_0 \geq 0$, the case $h^4_0 = 0$ taking place if and only if the initial density is independent of coordinate.

(ii) When $A(y)$ is a diagonal matrix, we have $h^2_1 = 0$. In addition, $\tilde{h}^2_2 = 0$ if and only if $\rho A = \text{const}$. Hence, when $\rho A = \text{const}$, the averaged (accurate to $O(\varepsilon^4)$) equation does not contain higher derivatives. Therefore, the dispersion of waves is absent. It can be shown [18] that, at $\rho A = \text{const}$, the dispersion is absent for any approximation in ε. This agrees with the fact that, as is known from acoustics of liquids, at $\rho A = \text{const}$, no refraction of waves occurs at the layer boundaries.

(iii) When $A(y)$ is not necessarily diagonal but $h^2_1 = 0$, the averaged equation (6) has the form

$$L^A\mathbf{v} = -\hat{\rho}\frac{\partial^2 \mathbf{v}}{\partial t^2} + \hat{A}\frac{\partial^2 \mathbf{v}}{\partial x^2} + \varepsilon^2 \tilde{h}^2_2 \frac{\partial^4 \mathbf{v}}{\partial t^2 \partial x^2} = 0. \tag{7}$$

If the condition $h^2_1 = 0$ is satisfied, it can be shown that $\tilde{h}^2_2 \geq 0$.

When $h^2_1 = 0$ (and, therefore, $\tilde{h}^2_2 \geq 0$), for long harmonic waves that propagate in the layered medium across the layers and are described by the equation $L^A\mathbf{v} = 0$, the velocity does not increase with frequency. Indeed, in searching for the solution to Eq. (7) in the form of a traveling harmonic wave $\mathbf{v} = e^{i(kx - \omega t)}\mathbf{e}$, we arrive at the eigenvalue problem,

$$(\hat{\rho}\omega^2 E - k^2\hat{A} + \varepsilon^2\omega^2 k^2 \tilde{h}^2_2)\mathbf{e} = 0$$
$$\text{or } (\hat{\rho}c^2 E - \hat{A} + p^2\tilde{h}^2_2)\mathbf{e} = 0,$$

where $p = \varepsilon\omega$ and $c = \dfrac{\omega}{k}$ is the phase velocity of waves.

Hence, if $\tilde{h}^2_2 > 0$, we have $c = c(p)$; i.e., a dispersion of waves takes place and $c(p)$ decreases with increasing p.

The condition $h_1^2 = 0$ is not too restrictive. In particular, it is satisfied in the following cases;

(i) acoustic waves in layered liquids; in this case, $\mathbf{v}$ is a scalar;

(ii) waves in locally orthotropic layered media with a symmetry plane parallel to the layers; in particular, in locally isotropic media (the matrix A is diagonal); and

(iii) waves in media with an arbitrary local anisotropy under the condition that $\rho(y)$ and $A(y)$ are even functions of y; in particular, such media are those consisting of the set of two repeated homogeneous layers with an arbitrary anisotropy of the layers.

Now, let us consider the waves propagating in an arbitrary direction [19]. Let these be plane waves propagating in the direction θ_1, θ_2, θ_3:

$$\mathbf{v} = \mathbf{v}(t, x), \quad x = \theta_j x_j, \quad \sum_{i=1}^{3} \theta_i^2 = 1.$$

Then, we reduce the averaged equation accurate to ε^4 in the form B

$$-\langle\rho\rangle\frac{\partial^2 \mathbf{v}}{\partial t^2} + \sum_{2 \le m \le 4} \varepsilon^{m-2} h_{l_1 l_2 l_3} \frac{\partial^m \mathbf{v}}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}} = 0$$

to the form

$$-\langle\rho\rangle\frac{\partial^2 \mathbf{v}}{\partial t^2} + H_2\frac{\partial^2 \mathbf{v}}{\partial x^2} + \varepsilon H_3\frac{\partial^3 \mathbf{w}}{\partial x^3} + \varepsilon^2 H_4\frac{\partial^4 \mathbf{v}}{\partial x^4} = 0, \quad (8)$$

where

$$H_l(\theta_1, \theta_2, \theta_3) = \sum_{l_1 + l_2 + l_3 = l} h_{l_1 l_2 l_3} \theta_1^{l_1} \theta_2^{l_2} \theta_3^{l_3}.$$

Note that, if $\rho \equiv 0$, $A_{ij} \equiv 0\ \forall i, j$ in the interval $a < y_1 < 1$ (where $0 < a < 1$) and $\theta_1 = 0$, the averaged equations describe the wave propagation along an infinite thin plate $0 < x_1 < a\varepsilon$ under the condition that the plate surfaces are free from any load. The higher-order equations for plates with allowance for a surface load were obtained in [21].

It was shown that the coefficients of Eq. (8) have the following properties.

(i) The matrices H_2 and H_4 are symmetric, whereas H_3 are antisymmetric $\forall\ \theta_j$.

(ii) For an unbounded medium, $H_2 > 0\ \forall\ \theta_j$, and, for plates, $H_2 \ge 0$.

(iii) $H_3 = 0\ \forall\ \theta_j$ in an unbounded medium if $\rho(y_1)$ and $A_{ij}(y_1)$ are even with respect to the plane $y_1 = 0.5$. For plates, $H_3(0, \theta_2, \theta_3) = 0$ under the condition of evenness with respect to the median plane. In particular, the evenness condition is satisfied for homogeneous plates with arbitrary anisotropy and for media consisting of a repeated set of two arbitrary layers.

(iv) The term $H_3\frac{\partial^3 \mathbf{v}}{\partial x^3}$ is usually present in the equations if the evenness condition fails. For example, $H_3 \neq 0$ in the general case of waves propagating along the layers in three-layer media or in two-layer plates.

Let us consider the solution to Eq. (8) in the form of harmonic waves

$$\mathbf{w}(t, x) = e^{i(kx - \omega t)}\mathbf{e}.$$

We obtain the equality

$$(-c^2(p)\langle\rho\rangle E + H_2 + ipH_3 - p^2 H_4)\mathbf{e} = 0. \quad (9)$$

Here, $c(p) = \frac{\omega}{k}$ is the wave propagation velocity and $p = \varepsilon k$. For all the real values of p, the matrix $H_2 + ipH_3 - p^2 H_4$ is Hermitian and, hence, its eigenvalues $\langle\rho\rangle c^2(p)$ are real.

Let μ_m and $\mathbf{e}_m$ be the eigenvalues and the corresponding eigenvectors of the matrix H_2. For simple eigenvalues μ_m, the following relations are valid [20]:

$$\langle\rho\rangle c_m^2(p) = \mu_m + g_m p^2 + O(p^4),$$
$$g_m = -(H_4\mathbf{e}_m, \mathbf{e}_m) + \sum_{l \neq m} \frac{1}{\mu_m - \mu_l}(H_3\mathbf{e}_m, \mathbf{e}_l)^2. \quad (10)$$

The behavior of the wave velocity with varying wave frequency depends on the sign of $g_m \neq 0$. From Eq. (10) it follows that the dispersion of waves is equally determined by the terms with the third and fourth derivatives of the displacements with respect to coordinates.

NUMERICAL STUDY OF AVERAGED EQUATIONS FOR STRATIFIED MEDIA AND PLATES

To calculate the coefficients of the averaged equations for unbounded media and plates consisting of homogeneous and, in the general case, anisotropic layers, a program was designed that, for the preset θ_1, θ_2, θ_3, calculated the matrices H_k and then the quantities μ_m and g_m. The following structures were considered [20]:

(A) isotropic layers;

(B) layers with cubic symmetry;

(C) orthotropic layers; and

(D) layers with arbitrary anisotropy.

In the cases (B) and (C), the directions of the symmetry axes were arbitrary. The coefficients of the equations corresponding to each of the media were chosen at random, and, in each of the cases (A), (B), and (C), no less than 50000 tests and, in the case (D), no less than 100000 tests were performed. Media and plates with numbers of layers up to 11 were considered.

The calculations showed that the appearance of the term on the order of ε (with nonzero skew-symmetric matrix H_3) is typical of the cases in which the evenness condition for $\rho(y)$ and $A_{ij}(y)$ fails. In the equations for the wave propagation along the layers ($\theta_2 = 1$) in a three-layer medium, in the general case we have $H_3 \neq 0$. In particular, $H_3 \neq 0$ for a locally isotropic three-layer medium if the values of the density ρ and the Lame coefficients λ are the same for all the layers while the values of μ are different.

The system of equations describing the wave propagation in the direction orthogonal to the layers ($\theta_1 = 1$) in an arbitrary locally isotropic stratified medium falls into a system of scalar equations. Then, $H_3 = 0$. However, if anisotropic layers are present, in the general case we have $H_3 \neq 0$. In particular, $H_3 \neq 0$ for a three-layer medium if its two layers are isotropic while the third layer consists of a material with cubic symmetry and none of its symmetry axes coincides with the x_1 axis.

In the general case of a plate consisting of two homogeneous layers (when the waves propagate along the layers), we have $H_3 \neq 0$.

It is of interest to consider the behavior of the velocities of harmonic waves with varying wave frequency. The calculations for both media and plates consisting of a periodic set of homogeneous layers showed that, for different structures, the following types of velocity variation with increasing frequency are possible:

(i) all the three velocities decrease;

(ii) two velocities decrease and one increases;

(iii) one velocity decreases and two increase;

(iv) all the three velocities increase; and

(v) one velocity decreases, one increases, and one is independent of frequency.

For waves propagating orthogonally to the layers, it was analytically shown [18] that $H_3(1, 0, 0) = 0$ and $H_4(1, 0, 0) \geq 0$ if the matrix A_{11} is diagonal or the medium is a two-layer one. Then, the velocities of harmonic waves determined from Eq. (9) do not increase with increasing frequency. A numerical study of more than 50 000 variants with numbers of layers up to 11 showed that, for waves propagating orthogonally to the layers in arbitrary multilayer media, the first of the listed types (type *1*) is always realized (a negative dispersion).

For waves propagating along the layers, the following results were obtained [20].

(i) In multilayer locally isotropic media (structure A), variants *1* and *2* were realized, and, in the presence of anisotropic layers (structures B, C, and D), variants *1*, *2*, and *3* took place. Thus, in layered media, at least one of the waves always exhibits a negative dispersion.

(ii) In one-layer isotropic plates (structure A), variant *5* was realized, and in anisotropic plates, variant *2* or variant *3* occurred.

(iii) In multilayer locally isotropic plates (structure A), variant *2* was realized; in the presence of anisotropic layers, variants *2* and *3* took place; and in the cases C and D, variant *4* was realized with a statistical frequency of 0.2% (all three waves had a positive dispersion).

Now, let us consider the characteristic features of wave propagation in plates. In an isotropic one-layer plate, oscillations in the directions x_1, x_2, and x_3 occur independently, whereas, in a two-layer plate, oscillations in the directions across the plate and along the wave propagation direction are coupled by the terms on the order of ε. In the absence of local isotropy (structures B, C, and D) in a one-layer plate, oscillations in the directions x_2 and x_3 are coupled in the principle terms; for a two-layer plate, an additional coupling on the order of ε occurs between oscillations in all the directions.

WAVES IN BARS

Let us consider the wave propagation in a bar that is inhomogeneous in thickness and anisotropic. The typical thickness of the bar is assumed to be much smaller than the wavelength [22, 23]. Let the x_1 axis be directed along the axis of the bar. We introduce the notation

$$a = \sqrt{\frac{\langle 1 \rangle}{\langle y_2^2 + y_3^2 \rangle}}.$$

The asymptotic expansion of the solution has the form [25]

$$\mathbf{u} \sim \sum_{q,\, n = 0}^{\infty} \varepsilon^{q+n} N_n^q(y_2, y_3) \frac{\partial^{q+n} \mathbf{v}}{\partial t^q \partial x_1^n},$$

where $\mathbf{v} = \mathbf{v}(t, x_1)$ is a four-dimensional column vector the elements of which are functions with a characteristic scale of variation much greater than ε; $N_n^q(y_2, y_3)$ are 3×4 matrices: $N_n^q = 0$ when $n < 0$ or $q < 0$, and

$$N_0^0 = \Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -ay_3 \\ 0 & 0 & 1 & ay_2 \end{pmatrix}.$$

The four components of the vector $\mathbf{v}$ have the meanings of displacement along the bar (longitudinal waves), displacements in two directions perpendicular to the bar axis (transverse or flexural waves), and the rotation angle of the bar cross section (torsional waves), respectively. The matrix Φ determines the rotation of the bar as a solid body.

We denote x_1 by x. The averaged equation with an infinite order of accuracy in ε for a bar the surface of which is free from loads has the form

$$\bar{L}\mathbf{v} \sim \sum_{2 \le q+n} \varepsilon^{q+n-2} h_n^q \frac{\partial^{q+n}\mathbf{v}}{\partial t^q \partial x^n} \sim 0. \qquad (11)$$

In this case, $N_n^q = 0$ and $h_n^q = 0$ for odd values of q. This averaged equation differs from the equations obtained for a layered medium and a plate in that the solution $\mathbf{v}$ has the fourth component, which characterizes the torsion of the bar. Higher-order equations for long wave propagation in homogeneous isotropic bars were proposed by different authors (see the review [26]). In the present paper, the equations for inhomogeneous anisotropic bars are derived and studied.

The derivation of the averaged equation is similar to the derivation of the equations for stratified media and plates but with allowance for the additional degree of freedom of the bar. In the case of discontinuous coefficients of the initial equation, instead of satisfying the differential equations for N_n^q, it is necessary to satisfy the corresponding integral identities. The averaged equations with allowance for a surface load were derived in [23].

In Eq. (11),

$$h_0^2 = \begin{pmatrix} \langle\rho\rangle & 0 & 0 & 0 \\ 0 & \langle\rho\rangle & 0 & -\langle\rho y_3\rangle \\ 0 & 0 & \langle\rho\rangle & \langle\rho u_2\rangle \\ 0 & -\langle\rho y_3\rangle & \langle\rho y_2\rangle & \langle\rho(y_2^2+y_3^2)\rangle \end{pmatrix}.$$

Now, for the matrix h_0^2, we use the notation $\hat{\rho}$. If the origin of coordinates is at the center of gravity of the bar cross section (this is assumed in what follows), the matrix $\hat{\rho}$ is diagonal.

Let us consider the averaged equation accurate to ε^3 by representing it in the form B:

$$L^B\mathbf{v} = -\hat{\rho}\frac{\partial^2\mathbf{v}}{\partial t^2} + H_2\frac{\partial^2\mathbf{v}}{\partial x^2} + \varepsilon H_3\frac{\partial^3\mathbf{v}}{\partial x^3} + \varepsilon^2 H_4\frac{\partial^4\mathbf{v}}{\partial x^4} = 0. \qquad (12)$$

Let us list the properties of averaged equation (12).

(i) The matrices H_2 and H_4 are symmetric, and the matrix H_3 is antisymmetric.

(ii) The matrix H_2 has a double zero eigenvalue: $\mu_2 = \mu_3 = 0$. Its other two eigenvalues, μ_1 and μ_4, are positive.

(iii) The equalities $(H_3)_{23} = (H_3)_{32} = 0$ are satisfied.

(iv) $H_3 = 0$ for a homogeneous isotropic bar; for inhomogeneous bars, $H_3 \ne 0$.

(v) In the case of a locally isotropic medium, the element $(H_2)_{44}$ responsible for the velocity of torsional wave propagation does not depend on the value of the Lame coefficient λ.

When studying the type of dispersion, we consider the solutions in the form of a traveling harmonic wave $\mathbf{v}(t, x) = e^{i(kx-\omega t)}\mathbf{e}$. Substituting this expression in the equation, we obtain

$$(-c^2(p)\hat{\rho} + H_2 + ipH_3 - p^2H_4)\mathbf{e} = 0,$$
$$c(p) = \omega/k, \quad p = k\varepsilon.$$

Multiplying this equation by $\hat{\rho}^{-1/2}$ and introducing $\mathbf{e}'$ according to the formula $\mathbf{e} = \hat{\rho}^{-1/2}\mathbf{e}'$, we arrive at the eigenvalue problem

$$(\hat{H}_2 + ip\hat{H}_3 - p^2\hat{H}_4)\mathbf{e}' = c^2(p)\mathbf{e}' \qquad (13)$$

with $\hat{H}_k = \hat{\rho}^{-1/2}H_k\hat{\rho}^{-1/2}$. Let μ_k ($k = 1, ..., 4$) be the eigenvalues of the matrix $\hat{H}_2$ and $\mathbf{e}_k$ be the corresponding eigenvectors forming an orthonormal system. If μ_k is a simple eigenvalue of the matrix $\hat{H}_2$, the corresponding eigenvalue $c_k^2(p)$ of problem (13) has the form

$$c_k^2(p) = \mu_k + g_kp^2 + O(p^4),$$

$$g_k = -(\hat{H}_4\mathbf{e}_k, \mathbf{e}_k) + \sum_{l \ne k}\frac{1}{\mu_k - \mu_l}(\hat{H}_3\mathbf{e}_k, \mathbf{e}_l)^2.$$

From properties *2* and *3* of the coefficients of the averaged equation, it follows that, if $\mathbf{e}_2$ and $\mathbf{e}_3$ are the eigenvectors corresponding to the eigenvalue $\mu_2 = \mu_3 = 0$, we have $(\hat{H}_3\mathbf{e}_3, \mathbf{e}_2) = 0$. Then, for the corresponding values of $c^2(p)$, we have $c^2(p) = \mu_k + g_kp^2 + O(p^3)$, where the values of g_k are calculated from certain more complicated formulas, as compared to those for simple eigenvalues.

NUMERICAL STUDY OF AVERAGED EQUATIONS OF WAVE PROPAGATION IN BARS

A program was designed, and the properties of the coefficients of the averaged equation and the frequency dependences of wave velocities were studied for bars with a rectangular cross section $|y_2|$, $|y_3| \le 1/2$; the bars were assumed to be homogeneous in length and consisting of two or four parts with different random values of density and elastic coefficients and with different anisotropies [22]. Different symmetry variants were considered for the constituent materials of the bars; the variants were the same as those considered above for waves in layered media and plates. The calculations were performed for several tens of different variants. The matrices H_k and the values of μ_k and g_k were calculated.

The results of these calculations showed that, in the case of waves propagating along the bar, the appearance of the term on the order of ε (with the nonzero skew-symmetric matrix H_3) was typical. For all the structures studied, the values of g_2 and g_3 were positive, $g_2 > 0$ and $g_3 > 0$, as in the case of a homogeneous isotropic bar; i.e., flexural waves always exhibited a positive dispersion. In approximately half of the cases, $g_4 < 0$, and, in half of the cases, $g_4 > 0$; i.e., torsional waves could exhibit a positive or negative dispersion, depending on the structure of the bar. It was found that $g_1 > 0$ in approximately one out of ten cases; longitudinal waves, as a rule, had a negative dispersion. No cases with $g_1 > 0$ and $g_4 < 0$ were observed.

As the simplest example of cases with all $g_k > 0$, a case was found where the bar consisted of two halves and the material of the halves had a cubic symmetry; also, an example may be the case of a bar consisting of four component bars, three of them being made from the same isotropic material and the fourth being also made from an isotropic material but with other values of elastic moduli.

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