ISSN 1063-7710, Acoustical Physics, 2010, Vol. 56, No. 6, pp. 893–903. © Pleiades Publishing, Ltd., 2010.

Transmission of Oscillations through a Layer of a Nonlinear Elastic Medium

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Received March 17, 2010

Abstract—The transmission of shear one-dimensional periodic perturbations through a layer of a nonlinearly elastic medium under the conditions close to resonance is considered. The layer separates two half-spaces consisting of a medium that is much more rigid, as compared to the medium in the layer. A system of differential equations is obtained for describing the slow variations in the amplitude and waveform of nonlinear strain and stress oscillations at the fixed boundary that occur because of the nonlinear properties of the medium while the other boundary performs arbitrary periodic motions in its plane. The period of these oscillations is close to the period of natural oscillations of the layer. It is shown that, in addition to continuous strain variations at the fixed boundary, strain variations containing strong discontinuities are possible. Relations at the discontinuities are obtained. The analogy between the equations derived for the case under study and the equations describing the propagation of strain waves in a homogeneous anisotropic elastic medium is pointed out.

DOI: 10.1134/S1063771010060138

Near-resonance steady-state oscillations have been studied in detail for a gas in a pipe with different conditions at its ends (see, e.g., [1–6]). Equations describing the establishment of periodic gas oscillations were obtained in [4]. Transverse oscillations in a layer of an isotropic medium were studied in [7, 8] for the case where the transverse oscillations were performed by one stress component and the mean stress value was zero. These oscillations are described by the equations with cubic nonlinearity, in contrast to the quadratic nonlinearity in the case of gas oscillations. In [9], planar motions were considered in the case of plasma layer oscillations in a magnetic field orthogonal to the layer under the particular assumption that the velocity of sound coincides with the van Alphen velocity. This leads to a complex resonance interaction of transverse Alphen perturbations and longitudinal acoustic waves.

In the case of small nonlinear oscillations in a layer of a weakly anisotropic elastic medium, which are considered below, two types of shear waves propagate in each of the directions with velocities differing by the quantity that depends on both nonlinearity and anisotropy. Thus, close-to-resonance conditions simultaneously occur for both types of waves. The presence of anisotropy in the medium means that one should study oscillations at which the medium performs arbitrary motions in planes orthogonal to the direction of wave propagation. It is assumed that no resonances occur between the longitudinal and transverse perturbations. In this case, longitudinal perturbations should not noticeably develop and their variations can be neglected.

INITIAL EQUATIONS

We consider a layer of a homogeneous elastic medium with a width L; the layer lies between two parallel planes orthogonal to a certain direction, which is denoted as the $x_3 = x$ axis of the Cartesian Lagrangian system of the initial state. The x_1 and x_2 axes lie in the plane that is parallel to the boundaries. We consider one-dimensional shear plane waves propagating in the x direction. The wave amplitude is assumed to be small, on the order of ε, and the medium is weakly anisotropic. According to [11], the variation of the longitudinal strain component in these waves is an order of magnitude (in ε) smaller and, if necessary, can be expressed through the amplitudes of shear waves. This allows us, in the statement of the problem, to consider the amplitude variations of only two shear waves.

Equations in terms of Lagrangian variables with allowance for nonlinearity and anisotropy in the principal terms have the form [10, 11]

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial u_i}\right), \quad \frac{\partial u_i}{\partial t} = \frac{\partial v_i}{\partial x}, \quad i = 1, 2, \qquad (1)$$

$$\Phi = \frac{f}{2}(u_1^2 + u_2^2) + \frac{g}{2}(u_2^2 - u_1^2) - \frac{\kappa}{4}(u_1^2 + u_2^2)^2,$$
$$v_i = \frac{\partial w_i}{\partial t}, \quad u_i = \frac{\partial w_i}{\partial x}. \tag{2}$$

Here, w_i is the displacement of the particles of the medium, u_i characterizes the shear strain of the planes parallel to the wave front, and v_i are the velocity components of the particles of the medium. The medium is assumed to be homogeneous, and its density is $\rho_0 =$ const; hence, in Eqs. (1) and in the subsequent calculations, for simplicity we set $\rho_0 = 1$. The function Φ is the elastic potential of a unit mass; f, g, and κ are the elastic coefficients of the medium. The coefficient f involved in the first term little differs from the shear modulus μ and is proportional to the squared velocity of small perturbations in the linear isotropic medium c_0^2, so that, taking into account that we have set $\rho_0 = 1$, we obtain $c_0^2 = f$. The last term in expression (2) for the elastic potential represents the nonlinear properties of the medium; the coefficient κ is finite and may have any sign. The factor g of the anisotropic term is assumed to be small and positive. For the nonlinear and anisotropic effects to be of the same order of magnitude, it is necessary that the coefficient g is on the order of ε^2. This condition is assumed to be satisfied in the subsequent consideration.

For Eqs. (1), we set the following boundary conditions: $v_i = 0$ at $x = 0$ and $v_i = \psi_i(t)$ at $x = L$, where the functions $\psi_i(t)$ are periodic or close to periodic ones with a period T and with amplitudes much smaller (at least two orders of magnitude in ε) than the amplitudes of the strain oscillations under study. We will study the behavior of the functions $u_1(0, t)$ and $u_2(0, t)$ representing the shear strains at the fixed boundary $x = 0$. The shear stresses at this boundary are expressed through the strains: $\sigma_i = \partial\Phi(u_1, u_2)/\partial u_i$.

It should be noted that, for small-amplitude waves propagating along a homogeneous background in a single direction (in the positive or negative direction of the x axis), the system of four equations can be approximately (but without loss of accuracy) transformed to a system of two equations with the potential Φ_1, which has the same structure as potential (2) with somewhat different coefficients $f \to f_1$, $g \to g_1$, and $\kappa \to \kappa_1$:

$$\frac{\partial u_i}{\partial t} \pm \frac{\partial}{\partial x}\left(\frac{\partial \Phi_1}{\partial u_i}\right) = 0, \quad i = 1, 2,$$
$$f_1 = \sqrt{f}, \quad g_1 = g/(2\sqrt{f}), \quad \kappa_1 = \kappa/(2\sqrt{f}). \tag{3}$$

The upper sign in Eqs. (3) corresponds to the motion of perturbations in the positive direction of the x axis, and the lower sign corresponds to the motion in the negative direction.

We make an assumption concerning the order of magnitude of some of the quantities involved in the statement of the problem. Since the amplitude of the emitted perturbations is taken to be $u_i \sim \varepsilon \ll 1$, these perturbations in the rough approximation can be described by the solutions to linearized equations (1).

In the approximation linear in ε, in the layer of the isotropic elastic medium between the fixed planes $x = 0$ and $x = L$, periodic natural oscillations may occur that are related to the propagation of linear perturbations along the x axis in both directions with the velocity $c_0 = \sqrt{f}$ and period $T_0 = 2L/c_0$. The effects of anisotropy and nonlinearity manifest themselves in that the velocities of the characteristics of two shear waves differ from c_0 by the quantities on the order of ε^2. Correspondingly, the travel times of these characteristics on the segment $[0, L]$ in both directions differ from T_0 by a quantity on the order of ε^2, while the external actions $\psi_i(t)$ are also small compared to ε.

We assume that the period of external actions T is constant and close to the period of natural oscillations T_0, so that $T - T_0 \sim \varepsilon^2$ and, hence, the mode of oscillation is close to the resonance one. In this case, it is useful to introduce the quantity $a = 2L/T =$ const with the velocity dimension. The quantity a slightly differs from the velocity of linear natural oscillations c_0, and this difference is $a - c_0 \sim \varepsilon^2$. Below, it will be shown that, if ψ_i is on the order of ε^3, the effect of external actions can compensate the effect of the nonlinear terms and, hence, steady-state periodic oscillations of the medium are possible.

Under the above assumptions, taking into account only the terms on the order of ε, the solution within one period can be constructed as a sum of linear waves propagating with the velocity a and reflected from the fixed boundaries. The corrections due to the nonlinear terms and the terms representing the anisotropy, as well as the corrections due to the boundary conditions, are taken into account in the next approximation. They lead to a slow variation of the waves from one period to another. To verify these considerations, we transform system of equations (1) by replacing the elastic potential Φ with the function

$$F(u_1, u_2) = \Phi - a^2\frac{u_1^2 + u_2^2}{2}.$$

Since a^2 differs from the coefficient f involved in Eq. (2) by a quantity on the order of ε^2, all the terms of the new function F, including those quadratic in u_i, are on the order of ε^4. We write the first group of Eqs. (1) so that all the terms on the order of ε are on the left-hand sides of the equations, while all the terms of higher order of smallness (they are on the order of ε^3) are on the right-hand sides:

$$\frac{\partial v_i}{\partial t} - a^2\frac{\partial u_i}{\partial x} = b_i, \quad b_i = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_i}\right),$$
$$\frac{\partial u_i}{\partial t} - \frac{\partial v_i}{\partial x} = 0; \quad i = 1, 2. \tag{4}$$

We reduce the left-hand sides of this system of four equations to the characteristic form

$$\frac{d_+ I_i^+}{dt} = b_i(I_k^+, I_k^-), \quad \frac{d_+}{dt} = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x},$$
$$\frac{d_- I_i^-}{dt} = b_i(I_k^+, I_k^-), \quad \frac{d_-}{dt} = \frac{\partial}{\partial t} - a\frac{\partial}{\partial x}, \quad i, k = 1, 2. \tag{5}$$

The new functions $I_i^+ = v_i - au_i$ and $I_i^- = v_i + au_i$ are the Riemann invariants of the left-hand sides of Eqs. (4). The initial variables are expressed through them as

$$u_k = \frac{1}{2a}(I_k^- - I_k^+), \quad v_k = \frac{1}{2}(I_k^- + I_k^+). \tag{6}$$

For the new functions I_k, the boundary conditions of the problem take the form

$$(I_i^+ + I_i^-)_{x=0} = 0, \quad (I_i^+ + I_i^-)_{x=L} = 2\psi_i(t).$$

THE LINEAR APPROXIMATION

To solve the problem, we use the method of successive approximations. The zeroth approximation is taken to be the periodic solution to the linear system with the right-hand sides of Eqs. (5) being neglected. The solution is represented in the form of waves traveling in the positive and negative directions of the x axis, in which the corresponding zeroth approximations of the Riemann invariants are retained:

$$I_i^{+0} = \varphi_i(at - x), \quad I_i^{-0} = \vartheta_i(at + x). \tag{7}$$

Then, for the strain and velocity components, the following expressions are valid in the linear approximation:

$$u_i^0 = \frac{1}{2a}[\vartheta_i(at + x) - \varphi_i(at - x)],$$
$$v_i^0 = \frac{1}{2}[\vartheta_i(at + x) + \varphi_i(at - x)]. \tag{8}$$

Since the external action $\psi_i(t)$ is assumed to have an amplitude much smaller than u_i, for the problem in the linear approximation, we should set zero boundary conditions at both boundaries: $v_i^0 = 0$ at $x = 0$ and at $x = L$. One of these conditions yields

$$\vartheta_k = -\varphi_k.$$

Still, it would be more convenient to retain both functions $\varphi_i(at - x)$ and $\vartheta_i(at + x)$ for their notations to indicate the structure of their arguments. The second boundary condition points to the fact that the functions φ_k and ϑ_k are periodic functions of their arguments with the period $2L$. As functions of t, they have the period T.

THE NONLINEAR APPROXIMATION

To obtain a nonlinear solution to Eqs. (5), we determine the next approximation.

Because of the aforementioned smallness of the functions $b_i(I_k^{\pm})$, on the right-hand sides of Eqs. (5), we use zeroth approximation (7) as the arguments of b_i.

With allowance for the form of dependence (7) of the functions φ_k and θ_k on x and t, in the expressions for $b_i(\varphi_k, \vartheta_k)$, the differentiation with respect to x can be replaced by differentiation with respect to t. Then, the functions b_i take the form

$$b_i = \frac{1}{2a^2}\left\{g_i - \frac{\kappa}{4a^2}[a(\varphi_i - \vartheta_i)^2 + (\varphi_{3-i} - \vartheta_{3-i})^2]\right\}$$
$$\times\left(\frac{\partial\varphi_i}{\partial t} + \frac{\partial\vartheta_i}{\partial t}\right) + (-1)^{3-i}\frac{\kappa}{4a^4}(\varphi_1 - \vartheta_1)(\varphi_2 - \vartheta_2) \tag{9}$$
$$\times\left(\frac{\partial\varphi_{3-i}}{\partial t} + \frac{\partial\vartheta_{3-i}}{\partial t}\right).$$

Here, we used the notations

$$g_1 = f - g - a^2, \quad g_2 = f + g - a^2.$$

Let us calculate the variation of the quantities $I_i^{\pm}$ within one period T at the left-hand boundary $x = 0$. For this purpose, we integrate Eqs. (5) with right-hand sides (9) along their characteristics.

Figure 1 shows the (x, t) characteristic plane with the boundaries of the elastic layer $x = 0$ and L, the characteristic $x - at = \text{const}$ (AD) running to the right with increasing time, and the characteristic $x + at = \text{const}$ (DB) running to the left. The functions $\varphi_i(at - x)$ and their derivatives are constant on the characteristics running to the right, and the functions $\vartheta_i(at + x)$ and their derivatives are constant on the characteristics running to the left.

We consider the variation of $I_i^{\pm}$ within a period at the left-hand boundary ($x = 0$). Taking the state of $I_i^{\pm}$ at the point A at $x = 0$ as the initial state and integrating Eqs. (5), we determine these functions at the point B at the same boundary within the period T with the boundary conditions being satisfied. For this purpose, we integrate the equations along the respective characteristics. The integration of the first of the equations

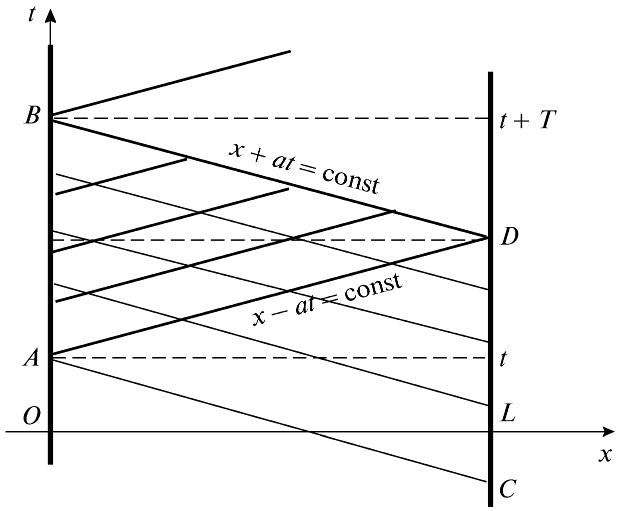

Fig. 1

along its characteristic AD yields the following result at the point $D(L, t+T/2)$ at the right-hand boundary:

$$\Gamma_i^+(D) = \Gamma_i^+(A) + \int_{AD} b_i dt.$$

In calculating the integrals of b_i, we take into account that the functions $\vartheta_i(at+x)$ vary along AD, but their values can be considered to be transferred without changes along their characteristics ($at + x =$ const) from the segment CD of the right–hand boundary; or, which is the same thing, from the segment AB of the boundary $x = 0$. Thus, when integrating the equation for Γ_i^+ along AD, the functions ϑ_i should be integrated from t to $t+T$ along the entire segment AB. As a result, we obtain

$$\Gamma_i^+(L, t+T/2) = \Gamma_i^+(L, t) + G_i T,$$
$$G_i = \int_{AD} b_i dt. \qquad (10)$$

The equations for the quantities G_i are obtained as a result of integration of the right-hand sides of Eqs. (5), after which the functions ϑ_k involved in them are replaced by ϑ_k according to the boundary condition of the linear problem $\vartheta_k = -\varphi_k$ at the fixed boundary $x = 0$. We obtain

$$G_i = \frac{1}{4a^2}\left\{g_i - \frac{\kappa}{4a^2}[3(\varphi_i + \bar{\varphi}_i)^2 + (\varphi_{3-i} + \bar{\varphi}_{3-i})^2\right.$$
$$\left. + 3h_{ii} + h_{(3-i)(3-i)}]\right\}\frac{\partial \varphi_i}{\partial t} \qquad (11)$$
$$-\frac{\kappa}{8a^4}(\varphi_1 + \bar{\varphi}_1)(\varphi_2 + \bar{\varphi}_2) + h_{12})\frac{\partial \varphi_{3-i}}{\partial t}, \quad i = 1, 2,$$

where

$$\bar{\varphi}_i = \frac{2}{T}\int_0^T \varphi_i dt, \quad h_{ij} = \bar{\varphi}_{ij} - \bar{\varphi}_i\bar{\varphi}_j,$$
$$\bar{\varphi}_{ij} = \frac{2}{T}\int_0^T \varphi_i\varphi_j dt, \quad i,j = 1, 2. \qquad (12)$$

The integration in Eqs. (12) is performed at $x = 0$.

The quantities determined by Eqs. (12) can slowly vary from one period to another, because, for every step of wave propagation from one boundary to the other, the role of the functions φ_i is played by their values obtained as a result of the preceding cycle.

In the same way, the integration is performed for the second equations (5) for Γ_i^- along the characteristic DB. Along this characteristic, $\vartheta_k(at+x) =$ const and the values of the functions φ_k are transferred along the characteristics $x - at =$ const from the segment AB of the t axis. The result of this integration yields

$$\Gamma_i^-(0, T) = \Gamma_i^-(L, t+T/2) - G_i T, \qquad (13)$$

since $\int_{DB} b_i dt = -G_i$.

Taking into account the boundary conditions of the nonlinear problem

$$\Gamma_i^-(0) = -\Gamma_i^+(0), \quad \Gamma_i^-(L) = -\Gamma_i^+(L) + 2\psi_i(t),$$

from Eqs. (14) and (13), we obtain

$$\Gamma_i^-(t+T) - \Gamma_i^-(t) = -2G_i T + 2\psi_i(t), \qquad (14)$$

$$\Gamma_i^+(t+T) - \Gamma_i^+(t) = 2G_i T - 2\psi_i(t). \qquad (15)$$

THE EQUATIONS OF THE SLOW EVOLUTION OF WAVES

As seen from Eqs. (14) and (15), the variations of the functions Γ_i^+ and Γ_i^- within the period are small (on the order of ε^3). Therefore, oscillations can be described on two time scales: within one period ($0 \le t \le T$), when the quantities rapidly vary with the characteristic time T, and small variation of the wave pattern from one period to another with a characteristic time of about T/ε^2 or greater. These variations are related to the variation of the "slow" time $0 < \tau < \infty$ characterizing the increase in the number of reflections [13, 14]; i.e., τ is associated with nT, where n is an integer. The term "slow" is determined by the slowness of the processes that occur with varying τ.

The variation of the wave pattern within a period represents a discrete function determined at times identical to an integral number of periods T. In the proposed approach, the discrete function is smoothed out and yields the dependence of all the quantities on the continuously varying variable τ; the small (about ε^2) variations of the functions within the period, after being divided by T, can be replaced by the derivatives with respect to τ. Since, in the real (unique) time, the instants $(t + T, \tau)$ and $(t, \tau + T)$ coincide, in the proposed approach, the values of all the functions at the ends of the segment $0 \le t \le T$ are considered to be coincident. Thus, the dependence of the solution on t at a fixed τ is determined on a closed curve with the length T.

Dividing each of Eqs. (14) and (15) by T, we represent them as a system of partial differential equations, where one of the variables is the slow time τ and the other is the real time t; these equations describe the variation of the functions $I_i^{\pm}$ at the fixed boundary of the layer ($x = 0$). As a result, we obtain

$$\frac{\partial I_i^-}{\partial \tau} = -2\left(G_i - \frac{1}{T}\psi_i\right), \quad \frac{\partial I_i^+}{\partial \tau} = 2\left(G_i - \frac{1}{T}\psi_i\right).$$

Now, we can return to the initial functions u_i involved in the statement of the problem. Using Eqs. (6), we obtain

$$\frac{\partial u_i}{\partial \tau} = -\frac{2}{a}\left(G_i - \frac{\psi_i}{T}\right). \tag{16}$$

In deriving Eq. (16), we took into account that $\partial\psi_i/\partial\tau = 0$, because we assumed that the functions ψ_i determining the velocity of oscillations at the boundary $x = L$ are periodic.

In these equations, we substitute Eqs. (11) for the functions G_i, which, as seen from Eqs. (11), are on the order of ε^3. Therefore, the error will be smaller than ε^3 if, in the expressions for G_i, the functions φ_k will be replaced by expressions in terms of u_i^0 according to Eqs. (8) with allowance for the fact that all the calculations are performed at $x = 0$, where $v_i = 0$, or are approximately replaced by the functions

$$\varphi_i(at) = -au_i(0, t).$$

In addition, to obtain the system of equations in a more conventional form, we replace the variable t by $\xi = at$. The variable ξ has the dimension of length and, within the period T, varies from 0 to $2L$. As a result, system of equations (16) takes the form

$$\frac{\partial u_i}{\partial \tau} - A_{ij}(u_k)\frac{\partial u_j}{\partial \xi} = \frac{\psi_i(t)}{L}, \qquad 2L = aT, \quad i, j, k = 1, 2. \tag{17}$$

The matrix $\|A_{ij}\|$ of the coefficients of Eqs. (17) proves to be symmetric, which points to hyperbolicity of the system of equations obtained above. For the coefficients A_{ij}, we obtain the expressions in the form of second partial derivatives of a certain function, so that $A_{ij} = \partial^2\bar{\Phi}/\partial u_i \partial u_j$.

The function $\bar{\Phi}$ has the form of a polynomial in powers of u_i, and the coefficients of this polynomial depend on the quantities obtained by averaging the solution over t within the period T; i.e., the coefficients are functions of the slow time τ:

$$\bar{\Phi} = \bar{f}\frac{u_1^2 + u_2^2}{2} + \bar{g}\frac{u_2^2 - u_1^2}{2} - \frac{\bar{\kappa}}{4}[(u_1 + \bar{u}_1)^2 + (u_2 + \bar{u}_2)^2]^2 - \bar{m}u_1u_2, \tag{18}$$

$$\bar{f} = \frac{f - a^2}{2a} - \bar{\kappa}(h_{11} + h_{22}),$$
$$\bar{g} = \frac{g}{2a} - \frac{\bar{\kappa}}{2}(h_{22} - h_{11}), \tag{19}$$
$$\bar{\kappa} = \frac{\kappa}{8a}, \quad \bar{m} = \frac{\kappa}{4a}h_{12},$$

$$\bar{u}_i = \frac{2}{T}\int_0^T u_i dt, \quad h_{ij} = \bar{u}_{ij} - \bar{u}_i\bar{u}_j, \quad \bar{u} = \frac{2}{T}\int_0^T u_i u_j dt.$$

As a result, system of equations (17) can be reduced to the form

$$\frac{\partial u_i}{\partial \tau} - \frac{\partial}{\partial \xi}\left(\frac{\partial \bar{\Phi}}{\partial u_i}\right) = \frac{\psi_i(\xi)}{L}. \tag{20}$$

The left-hand sides of Eqs. (20) are similar in form to Eqs. (3) for elastic waves traveling in the negative direction of the x axis over a boundless homogeneous background [12]. In this case, the function $\bar{\Phi}(u_i, \bar{u}_i, \bar{u}_{ik})$ plays the role of the elastic potential and its structure is similar to that of elastic potential (2) of the initial medium, with the only difference being that the coefficients of the new elastic potential depend on τ in the general case and are constant (as in Eq. (2)) only in the case of periodic oscillations. In addition, from Eqs. (19), it follows that all the coefficients of expansion (18), except for $\bar{\kappa}$, are on the order of ε^2 while the function itself is $\bar{\Phi} \sim \varepsilon^4$. It should also be noted that, if the initial elastic medium is isotropic and, in expansion (2), the coefficient is $g = 0$, the new elastic potential $\bar{\Phi}$ acquires an expansion term containing the anisotropy coefficient; i.e., the averaged properties of oscillations may lead to the appearance of anisotropy.

If we fix the averaged quantities involved in the expression for $\bar{\Phi}$ (they vary slowly), the resulting system of equations (17) or (20) will be a nonlinear hyperbolic system and, therefore, this allows both continuous solutions and solutions with discontinuities. It possesses two families of characteristics of fast ($\bar{c}_2$) and slow ($\bar{c}_1$) waves. The characteristic velocities $\bar{c}_1$ and $\bar{c}_2$ are determined through the function $\overline{\Phi}$ by the same formulas by which c_1 and c_2 are expressed through the function Φ and differ by the quantity $\bar{c}_2 - \bar{c}_1 \sim \varepsilon^2$. In this case, the quantities $\bar{c}_1$ and $\bar{c}_2$ themselves are on the order of ε^2 and, hence, in the course of the evolution of oscillations, may pass through zero and change sign, which leads to propagation of small perturbations with varying τ in both directions in the region of variation of the variable t.

In subsequent calculations, as the second variable in Eqs. (17) and (20), we use the variable t (which was used initially), rather than the variable $\xi = at$.

In [14], using the integral conservation laws of the nonlinear elasticity theory, we derived a system of integral equations for describing the behavior of the functions $u_i(0, t, \tau)$:

$$\frac{d}{d\tau}\int_{t_1}^{t_2} u_i dt - \frac{\partial\overline{\Phi}}{\partial u_i}\bigg|_{t=t_2} + \frac{\partial\overline{\Phi}}{\partial u_i}\bigg|_{t=t_1} = \frac{1}{L}\int_{t_1}^{t_2}\psi_i dt, \quad (21)$$

$$i = 1, 2.$$

Here, t_1 and t_2 are arbitrary values of time smaller than T. This system of equations shows that Eqs. (20) express the conservation laws (with allowance for external actions given by the functions $\psi_i(t)$). From Eqs. (21), we obtain both differential equations (20) in the case of smooth solutions and the relations at discontinuities

$$\left[\frac{\partial\overline{\Phi}}{\partial u_i}\right] = W[u_i], \quad i = 1, 2, \quad (22)$$

where $W = \frac{dt}{d\tau}$ is the "velocity" of the front of discontinuity and $\overline{\Phi}$ is the function determined by Eq. (18). In Eq. (22), it is assumed that the discontinuity occurs for u_i alone, whereas the averaged quantities (12) are continuous.

For discontinuous solutions to exist, the evolution conditions should be satisfied at the discontinuity front. In the problem of interaction between the front and small one-dimensional perturbations, the relations at the discontinuity should be sufficient to make it possible to unambiguously determine the perturbation of the discontinuity velocity W and the amplitudes of perturbations leaving the front. The evolution conditions consist in that the number of characteristics leaving the discontinuity should be one less than the number of relations at the discontinuity [15]. If a single system of equations describes the solution on both sides of the front and the number of relations at the discontinuity is identical to the order of the system, the evolution condition can be formulated as follows: on the characteristic plane, the characteristics of one family converge to the discontinuity, i.e., arrive at it on both sides, while the characteristics of other families intersect it, i.e., arrive at it on one side and leave it on the other side. In the given problem, the relations at the discontinuity are represented by two equalities (22), so that, for the evolutionary type of the discontinuity, it is necessary that only one characteristic leaves it and the other three arrive at it.

Now, let us consider some of the problems that can be stated for system of equations (20).

DETERMINATION OF THE NECESSARY EXTERNAL ACTIONS

The simplest statement of the problem for the evolution model consists in that it is necessary to determine an external action that would maintain a desired mode of variation of strain components with increasing number of reflections, i.e., that would provide a preset form of the functions $u_i(t, \tau)$. Knowing the functions $u_i(t, \tau)$, we can calculate the average quantities $\bar{u}_i(\tau)$ and $\bar{u}_{ij}(\tau)$ and then determine the function $\overline{\Phi}$. Substituting this function in system of equations (20), we can determine the functions $\psi_i(t, \tau)$ of the external action that is necessary for maintaining the given mode.

From Eqs. (20), one can see that, in the case in which it is necessary to maintain a periodic mode of oscillations, the required external action is very small: $\sim\varepsilon^3$; i.e., it is two orders of magnitude smaller than the amplitudes of oscillations themselves. The solution under consideration should be stationary in the slow time regime; i.e., the period average quantities $\bar{u}_i$ and $\bar{u}_{ij}$ do not depend on τ. Therefore, coefficients (19) of the function $\overline{\Phi}$ are also constant and this function only depends on the variables u_1 and u_2. For the external action components, from Eqs. (17), we obtain the expressions

$$\psi_i(t) = -\frac{L}{a}A_{ij}\frac{du_j}{dt}, \quad a = \frac{2L}{T}.$$

Let us illustrate this by using a specific example. We assume that it is necessary to determine the functions $\psi_1(t)$ and $\psi_2(t)$ to maintain oscillations of the following type:

$$u_1 = A\sin\omega t, \quad u_2 = B\cos\omega t, \quad (23)$$

where A and B are positive constants on the order of ε. Then,

$$\bar{u}_1 = \bar{u}_2 = \bar{u}_{12} = 0, \quad \bar{u}_{11} = A^2,$$

$$\bar{u}_{22} = B^2, \quad a = L\omega/\pi, \quad m = 0$$

and, for the functions $\psi_i(t)$, we obtain the expressions

$$\psi_1 = -\frac{A\pi}{2a^2}\left[g_1 - \frac{\kappa}{8}(9A^2 + B^2)\right]\cos\omega t$$
$$-\frac{3A\pi\kappa}{32a^2}(A^2 - B^2)\left(\cos\frac{\omega t}{2} + \cos\frac{3\omega t}{2}\right),$$
$$\psi_2 = -\frac{B\pi}{2a^2}\left[g_2 - \frac{\kappa}{8}(A^2 + 9B^2)\right]\sin\omega t$$
$$-\frac{3B\pi\kappa}{32a^2}(A^2 - B^2)\left(\sin\frac{\omega t}{2} - \sin\frac{3\omega t}{2}\right).$$

Remember that $g_1 = f + g - a^2$ and $g_2 = f - g - a^2$ and that the coefficient g characterizes the anisotropy of the initial medium and is on the order of ε^2; hence, g_1 and g_2 are also on the order of ε^2. The expressions for $\psi_i(t)$ show that these quantities should be on the order of ε^3.

When the initial medium is isotropic (i.e., $g = 0$), we have $g_1 = g_2 = \bar{g} = f - a^2$. If, in such a medium, we consider the solution with $B = A$, the formulas for ψ_i take the form

$$\psi_1 = -\alpha\cos\omega t, \quad \psi_2 = \alpha\sin\omega t,$$
$$\alpha = \frac{\pi}{2a^2}A\left(\bar{g} - \frac{5\kappa}{4}A^2\right). \tag{24}$$

Thus, for the strain components u_i at the boundary $x = 0$ to perform circular periodic oscillations with a constant amplitude A, it is necessary that the velocity vector of the other boundary ($x = L$) has the amplitude α and rotates with the frequency ω. From the expression for α, one can see that undamped oscillations are also possible when $\alpha = 0$, i.e., in the absence of external action. The amplitude of these oscillations is $A = \sqrt{4\bar{g}/5\kappa}$. At $\alpha = 0$, we also obtain the trivial solution $A = 0$. However, if the initial medium possesses a minor anisotropy $g \neq 0$, undamped oscillations are impossible in the absence of external actions ($\psi_i = 0$) (this statement is justified below).

When $\alpha > 0$, i.e., $A < \sqrt{4\bar{g}/5\kappa}$, from Eqs. (23) and (24), it follows that the external actions exhibit a phase advance of $\pi/2$ with respect to the oscillations of the vector $u_i(0, t)$; when $\alpha < 0$ and $A > \sqrt{4\bar{g}/5\kappa}$, the external actions have a phase lag with respect to $u_i(0, t)$.

If we consider the process of the establishment of oscillations under study, i.e., if, in Eqs. (23), we set $A = A(\tau)$ and $B \equiv A$, then Eqs. (20) acquire the terms $(\partial A/\partial\tau)u_i$ on their left-hand side, which will lead to the appearance of the terms $(\partial A/\partial\tau)Lu_i$ in the vector $\{\psi_i\}$; note that these terms are in phase or in antiphase with the vector $\{u_i\}$ (Eq. (23)), depending on the sign of $(\partial A/\partial\tau)$.

PERIODIC OSCILLATIONS OF THE LAYER

Let us now present the oscillation velocity components at one of the boundaries ($x = L$) in the form of periodic functions $\psi_i(t)$ with the period T. We intend to determine the stationary periodic solutions for the strain components at the other boundary ($x = 0$). For system of equations (20), we consider periodic solutions $u_i = u_i(t)$, i.e., solutions for which $\partial u_i/\partial\tau = 0$. The system of equations takes the form

$$\frac{d}{dt}\left(\frac{\partial\bar{\Phi}}{\partial u_i}\right) = -\frac{a}{L}\psi_i(t), \tag{25}$$

and it can be integrated over t. Since the functions $\psi_i(t)$ represent the velocity components v_i of the boundary $x = L$, the integrals of them with respect to t have the meaning of displacement w_i of this boundary and also are periodic functions of time. We denote these quantities as follows:

$$\frac{a}{L}\int\psi_i(t)dt = B_i(t).$$

Evidently, the functions $B_i(t)$ can be considered as continuous ones.

As a result of integration of Eq. (25), we obtain the system of equations

$$\frac{\partial\bar{\Phi}}{\partial u_i} = -B_i(t), \quad i = 1, 2. \tag{26}$$

Because of the stationary behavior, the period average quantities $\bar{u}_i$ and $\bar{u}_{ij}$ are constant. Then, the function $\bar{\Phi}$ depends on only two variables, u_1 and u_2, which characterize the strain at the current instant of time. In this case, polynomial (18) representing the function $\bar{\Phi} = \bar{\Phi}(u_1, u_2)$ has constant coefficients. This means that the equations of system (26) are algebraic at any instant of time t. The system is a fifth-order one, and, therefore, it may have one, three, or five real solutions. Each of the solutions represents a point on the (u_1, u_2) plane. We denote these points by S_α, where α is the number of solution. For illustration, we can represent system of equations (26) by means of an equation for u_1 and an expression for u_2 through u_1:

$$\frac{\kappa}{4}u_1^3\left[1 + \frac{B_2^2}{(2gu_1 + B_1)^2}\right] - (f - g - a^2)u_1 + \frac{B_1}{a} = 0,$$
$$u_2 = \frac{B_2 u_1}{2gu_1 + B_1}. \tag{27}$$

The solutions $u_i(t)$ to system of equations (26) are actually functions of B_i. The periodic functions $B_i(t)$ themselves can be represented as the coordinates of a point that performs periodic motions along a certain curve in the (B_1, B_2) plane (Fig. 2). This may be a closed curve or a segment periodically traveled in both

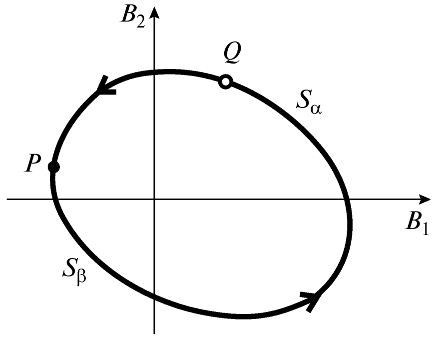

Fig. 2.

directions, which can also be considered as a version of a closed curve. At the same time, the points S_α describe certain curves in the plane (u_1, u_2).

The solution to Eqs. (26) may be represented by the coordinates u_i^α of any of the points S_α, if this point exists for any value of t.

Equations (26) (or (27)) necessarily have at least one real solution corresponding to a certain point S_1; if this solution exists for any t, it evidently is continuous and periodic. Equations (26) may simultaneously have solutions represented by some other points S_α in the (u_1, u_2) plane. In the presence of several points S_α, the solutions may have discontinuities, which correspond to jumps from one of the points to another in the (u_1, u_2) plane. In this case, Eqs. (22) will be satisfied, because, in the stationary case under study, $W = 0$.

The positions and types of the points S_α can be described by the function $N(u_1, u_2)$:

$$N(u_1, u_2) = \overline{\Phi}(u_1, u_2) + B_1(t)u_1 + B_2(t)u_2,$$

for which these points are stationary; i.e., they are points with zero partial derivatives with respect to the arguments of this function. Indeed, Eqs. (26) written with the use of the function $N(u_1, u_2)$ take the form

$$\frac{\partial N}{\partial u_i} = 0, \quad i = 1, 2. \tag{28}$$

Note that Eqs. (28) coincide with the equations that determine the positions of singular points of the system of equations used in studying stationary structures of shock waves in a medium with the elastic potential $\overline{\Phi}(u_1, u_2)$ [11, 16].

The types of stationary points of Eqs. (28) are determined by the signs of the eigenvalues of the matrix $\partial^2 N/\partial u_i \partial u_j$, which coincides with the matrix of the coefficients of Eqs. (17) $A_{ij} = \partial^2 N/\partial u_i \partial u_j$; i.e., by the signs of the characteristic values of the velocities $\bar{c}_1$ and $\bar{c}_2$ of system (17), which are determined by the equations

$$|A_{ij} - \bar{c}\delta_{ij}| = 0.$$

If both $\bar{c}_1$ and $\bar{c}_2$ are negative, the corresponding point u_i^α is the maximum of the function $N(u_1, u_2)$; if both eigenvalues $\bar{c}_1$ and $\bar{c}_2$ are positive, this point is the minimum; if the signs of $\bar{c}_1$ and $\bar{c}_2$ are different, the stationary point is a saddle point.

As $B_i(t)$ varies along its trajectory, stationary points may appear and disappear in the (B_1, B_2) plane (Fig. 2). As B_i varies, these points always appear and disappear in pairs at the instant when the curve that represents fifth-degree polynomial (27) touches the u_1 axis. These pairs are of two types. Some of them contain a saddle and a minimum, and other include a maximum and a saddle. In the general case (at certain values of $B_i(t)$), each of the pairs first (at the instant of its generation) appears in the form of a complex stationary point on the (u_1, u_2) plane and then splits into one of the aforementioned pairs of points. As $B_i(t)$ varies further, these points may disappear also in pairs by preliminarily merging into a single point. This process is illustrated in Fig. 3 for a medium with $\kappa > 0$ in the form of one of the possible sequences of patterns of lines corresponding to $N(u_1, u_2) = \text{const}$ with varying $B_i(t)$.

As it was noted above, along with continuous solutions to Eqs. (26), discontinuous solutions, which are represented in the (u_1, u_2) plane by a jump from one point S_α to another S_β, are possible. For a discontinuous solution to exist, certain requirements should be satisfied, in particular, the discontinuity must be evolutionary. In addition, because of periodicity, the solution should return to the initial branch S_α within a period. However, this is impossible by an inverse jump, because, if the jump in some direction of parameter variation is evolutionary, the jump in the opposite direction is nonevolutionary. Hence, the return to the branch S_α should occur along the trajectory of the continuous solution S_β and these two branches of the solution should have a common point. Such a point Q appears when two stationary points of the pair under consideration merge (Fig. 2). The coincidence of two points S_α and S_β in the stationary solution means the possibility of the existence of an infinitely weak immobile discontinuity; i.e., the zero value of the characteristic velocity. (The particular case of a simultaneous mergence of more than two points is not considered in this paper.)

For system of two equations (17) with two conservation laws, the evolutionary discontinuities (shock waves) are the discontinuities with one characteristic leaving the front. Among immobile shock waves ($W = 0$), these are fast shock waves corresponding to the jumps from the maximum of the function $N(u_1, u_2)$

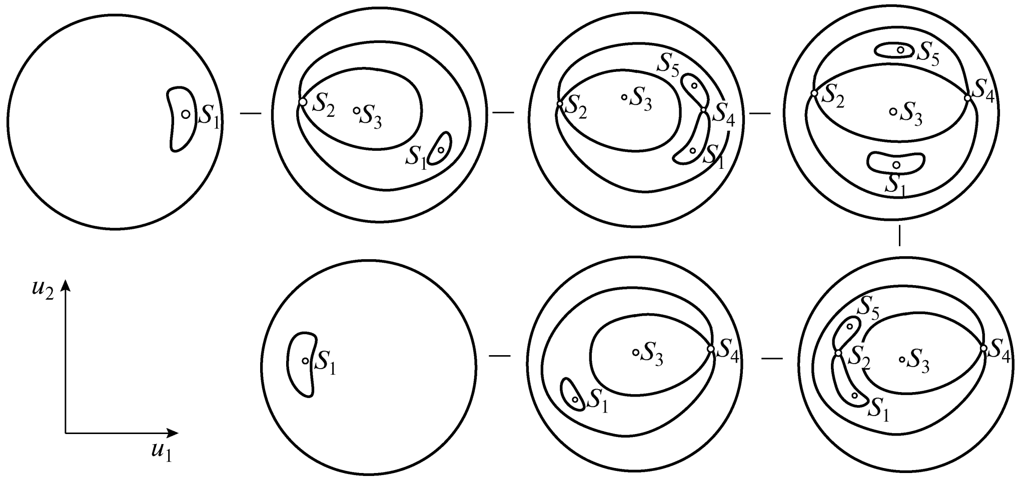

Fig. 3.

(the points S_1 and S_5 in Fig. 3) to the saddle (the points S_2 and S_4) and the slow shock waves corresponding to the jumps from the saddle to the minimum of the function $N(u_1, u_2)$ (the points S_3). As was mentioned above, a periodic solution $u_1(t)$, $u_2(t)$ containing the jump $S_\alpha \to S_\beta$ is possible if the curve $B_1(t)$, $B_2(t)$ in the (B_1, B_2) plane has a point at which S_α and S_β coincide. In Fig. 2, this point is represented by the empty dot Q. The point of this curve that corresponds to the jump from one branch of the solution to another is represented in Fig. 2 by the full dot P. Both solutions S_α and S_β should exist for all the values of t in the vicinities of these points.

Let, for example, the point $(B_1(t), B_2(t))$ perform oscillations along the segment of a certain curve and the point corresponding to the coincidence of S_α and S_β represent one of the ends of the segment, whose all other points correspond to the points S_α and S_β. In this case, the solution is partially represented by the coordinates $u_1^{(\alpha)}(t)$, $u_2^{(\alpha)}(t)$ of the point S_α and, partially, by the coordinates $u_1^{(\beta)}(t)$, $u_2^{(\beta)}(t)$ of the point S_β; at a certain value of t, the solution has a jump from the curve described in the plane (u_1, u_2) by the point S_α to the curve described by the point S_β. In the stationary solution, the position of the jump is arbitrary. At the same values of the function $B_i(t)$, the jump may be absent while the solution can be represented by any of the points S_α or S_β.

In Fig. 4, the solid lines qualitatively illustrate the variation of one of the functions u depending on the time t. The curves lying above the dashed line correspond to S_α, and the curves lying below the dashed line, to S_β. On the dashed line, the points S_α and S_β coincide, which, as was mentioned above, means a zero value of a certain characteristic velocity. The uppermost solid line corresponds to the continuous solution S_α. The lowest thick curve corresponds to the solution with a discontinuity. This solution contains the point Q, at which S_α coincides with S_β. In Fig. 2, this point is also denoted by the letter Q.

Solutions simultaneously containing the fast $S_\alpha \to S_\beta$ and slow $S_\beta \to S_\gamma$ shock waves within one period are possible. However, in this case, the curve representing the points $(B_1(t), B_2(t))$ in Fig. 2 should contain the points at which, in the (u_1, u_2) plane, the points S_γ and S_β and the points S_β and S_α coincide. The positions of the shock waves are arbitrary, as in the previous case.

Let us investigate the possibility of the existence of nonlinear undamped oscillations in the absence of external actions, i.e., at B_i = const. (For a specific case, this problem was discussed above.) If stationary points are isolated ones, which is the general case, we evidently have u_i = const, and oscillations are absent. An exception is the case where the surface determined by the function $N(u_1, u_2)$ has a circular symmetry, which occurs when, in the formula for $N(u_1, u_2)$, the coefficient is $g = 0$. This means the absence of wave anisotropy in the initial elastic medium. In this case, the surface representing the function $N(u_1, u_2)$ and

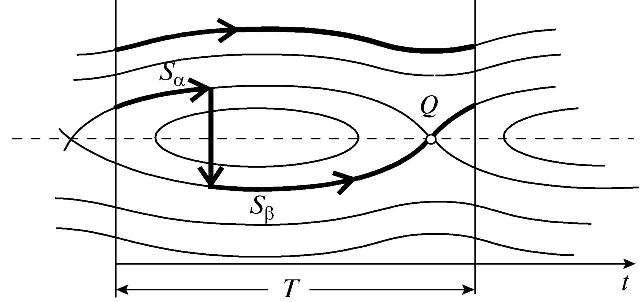

Fig. 4.

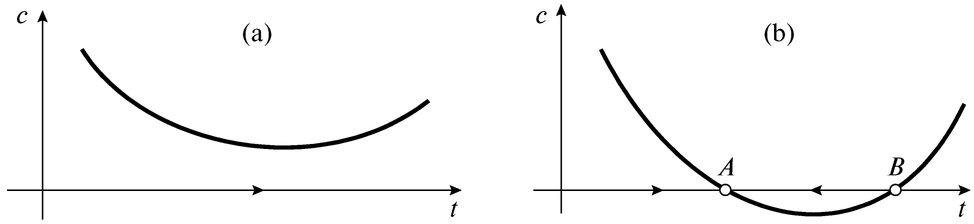

Fig. 5.

shaped as a volcano crater at $\kappa > 0$ reaches its maximal value at the circumference. All the circumference consists of stationary points of the function $N(u_1, u_2)$.

To verify that, in this case, oscillations are possible, i.e., variation of u_i without a change in B_i is possible, we first consider the situation wherein small violations of the circular symmetry of the function $N(u_1, u_2)$ take place. In this case, the height of the wall of the "volcano crater" little differs from constant. On the upper part of the crater surface near the circumference, four stationary points of the function $N(u_1, u_2)$ are positioned. Small variations of B_i are sufficient for these points to begin to move over the upper part of the crater surface, i.e., approximately over the circumference. In the limit where symmetry violations tend to zero, the motion of the point $(u_1(t), u_2(t))$ representing the solution is possible over the circumference without any changes in B_i, i.e., without external actions. The resulting solution evidently consists of circular waves, which are known to be undamped and undeformed in the course of propagation.

If the medium has even the slightest anisotropy $g \neq 0$, such a solution is impossible.

THE PROCESS OF DISCONTINUITY FORMATION

Let us discuss the process of discontinuity formation in the problem of layer oscillations under the assumption that this process is quasi-stationary. First, we consider the situation wherein a stationary solution containing an immobile evolutionary discontinuity is present. In this case, we have one family of characteristics with a characteristic velocity directed toward the point of discontinuity. At the discontinuity, this velocity exhibits a jump with a change of sign. The other characteristic velocity does not change sign at the discontinuity. The closed line with the length T, within which the variable t varies, should contain another point at which the velocities of the characteristics of this family are zero and the velocity changes sign. Evidently, this is the point discussed above, namely, the point at which two stationary points of the function N coincide. The characteristics of this family move in opposite directions from the aforementioned point to the discontinuity with the velocity $dx/dt = c$.

Therefore, the quasi-stationary process of discontinuity formation can be represented as follows. Initially, at a certain τ, both characteristic velocities do not change sign as t varies within the period (Fig. 5a). In the course of evolution of the solution under the effect of external actions, one of the characteristic velocities changes sign within one of the segments of the t axis. In this case, we obtain two points where this characteristic velocity is zero (Fig. 5b).

The signs of the characteristic velocity are always such that, at one of the points, the characteristics diverge (point B in Fig. 5b) and, at the other point, they converge (point A in Fig. 5b) in the course of their propagation along the t axis. At the point at which the characteristics converge (in Fig. 5, the direction of motion of the characteristics is indicated by arrows), a discontinuity appears, because the characteristics arriving from different sides carry different perturbations. The point at which the characteristics diverge serves as the source of characteristics arriving at the discontinuity from different sides.

The discontinuity formed in the course of the evolution of a nonstationary solution can disappear. For this to occur, the corresponding characteristic velocity should resume the same sign within the whole segment of t variation. This may occur if the discontinuity approaches the point from which the characteristics move away (in the case of the quasi-stationary process, this is the empty circle in Fig. 2); here, the magnitude of discontinuity becomes zero.

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research (projects nos. 08-01-00612 and 08-01-00401) and the Presidential Program in Support of Leading Scientific Schools of Russia (grant no. NSh-4810.2010.1).

REFERENCES

1. W. Chester, J. Fluid Mech. **18**, 44 (1964).
2. A. N. Kraĭko and A. L. Ni, Prikl. Mat. Mekh. **44**, 77 (1980).
3. A. L. Ni, Prikl. Mat. Mekh. **47**, 607 (1983).
4. V. É. Gusev, Akust. Zh. **30**, 204 (1984) [Sov. Phys. Acoust. **30**, 121 (1984).

5. S. A. Egorushkin, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 4, 107 (1984).
6. O. V. Rudenko, C. M. Hedberg, and I. O. Énflo, Akust. Zh. **47**, 525 (2001) [Acoust. Phys. **47**, 452 (2001)].
7. N. R. Sibgatullin, Prikl. Mat. Mekh. **36**, 79 (1972).
8. O. V. Rudenko, C. M. Hedberg, and I. O. Énflo, Akust. Zh. **53**, 522 (2007) [Acoust. Phys. **53**, 455 (2007)].
9. N. R. Sibgatullin, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 2, 131 (1976).
10. D. R. Bland, *Nonlinear Dinamic Elasticity* (Waltham, Toronto, 1969; Mir, Moscow, 1972).
11. A. G. Kulikovsky and E. I. Sveshnikova, *Nonlinear Waves in Elastic Media* (Mosk. litseĭ, Moscow, 1998) [in Russian].
12. A. G. Kulikovsky, Prikl. Mat. Mekh. **50**, 597 (1986).
13. A. G. Kulikovsky and E. I. Sveshnikova, Prikl. Mat. Mekh. **70**, 1040 (2006).
14. A. G. Kulikovsky and E. I. Sveshnikova, Prikl. Mat. Mekh. **72**, 985 (2008).
15. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics,* 2nd ed. (Nauka, Moscow, 1986; Pergamon, Oxford, 1987).
16. A. G. Kulikovsky and E. I. Sveshnikova, Prikl. Mat. Mekh. **51**, 926 (1987).

Translated by E. Golyamina