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CLASSICAL PROBLEMS OF LINEAR ACOUSTICS AND WAVE THEORY

The Effect of a Surface Impedance Load on the Behavior of Quasi-Rayleigh Waves near a Cylindrical Cavity[1]

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Abstract—The effect of a surface impedance load on the properties of axisymmetric quasi-Rayleigh waves propagating along the boundaries of a cylindrical cavity is investigated. By solving the problem by means of the impedance method, a dispersion equation for these waves is obtained. It is shown that the equation can be represented as the condition that the determinant of the sum of impedance matrices of the load and the medium is zero. Analysis of this equation allows one to investigate the effect of the surface load on the behavior of quasi-Rayleigh waves and on their critical frequencies. The conditions that should be met by the impedance load for quasi-Rayleigh waves to be absent near the cavity or for one or two such waves to exist are determined. The choice of the load is specified for the propagating quasi-Rayleigh wave to possess preset dispersion properties. The conclusions drawn on the basis of this study are illustrated by several examples of load models that can be implemented in practice.

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Surface waves in elastic isotropic media have been the subject of numerous publications. Reviews devoted to classification and practical application of such waves can be found in, e.g., [1, 2]. An important class of surface waves is formed by the Rayleigh-type waves propagating along plane or curved boundaries. For example, Biot [3] obtained the dispersion characteristics of axisymmetric quasi-Rayleigh waves and Stoneley waves near a cylindrical cavity. The behavior of waves with a more complex symmetry in the high-frequency approximation was considered in [1, 4]. It was shown that the phase and group velocities of these waves depend on the curvature radius of the surface. The problem of describing the quasi-Rayleigh waves at a plane boundary of an isotropic half-space with a two-component surface impedance load, as well as the problem of determining their dispersion characteristics, was studied in [5, 6]. In these publications, the load was represented by a diagonal impedance matrix describing the linear relation between the stress and velocity vectors. Publications [5–8] are examples of using the impedance method for determining the properties of different types of waves under specific types of loads (e.g., cracked or inhomogeneous layered media).

In the present paper, we study the properties of axisymmetric modes near an infinite cylindrical cavity in an elastic medium the surface of which is subjected to an impedance load. The latter is uniformly distributed and acts locally. The radial and tangential stresses caused by this load depend on each of the displacement components (normal and tangential ones). The load impedance matrix is assumed to be Hermitian.

The dispersion properties of quasi-Rayleigh waves near a cylindrical cavity in the absence of the impedance load at the boundary are well known [3]. We briefly describe the method of their determination, which allows us to give a natural interpretation of the solution to the problem formulated in the present paper. Let us consider harmonic waves with a frequency ω (the factor $e^{-i\omega t}$ is omitted below for brevity) that propagate near a cavity with a radius a. The elastic medium is characterized by the Lame constants λ and μ and by the density ρ. The wave number and the velocity of longitudinal waves are $k_l = \omega/c_l$ and $c_l = \sqrt{(\lambda + 2\mu)/\rho}$, respectively. For shear waves, the corresponding quantities are $k_t = \omega/c_t$ and $c_t = \sqrt{\mu/\rho}$. Taking into account the geometry of the problem, it is convenient to use the cylindrical coordinate system (r, θ, z) with the origin of coordinates at the axis of the cavity. It is known that the displacement vector can always be represented as [9]

$$\bar{u}(u_r, u_\theta, u_z) = \operatorname{div}\phi + \operatorname{curl}\bar{\psi},$$

where ϕ is the scalar potential and $\bar{\psi} = (\psi_r, \psi_\theta, \psi_z)$ is the vector potential. Since, in the axisymmetric case, the form of the solutions does not depend on the angle θ, the azimuth displacement u_θ and the derivatives $\partial\psi_r/\partial\theta$ and $\partial\psi_z/\partial\theta$ involved in the expressions for u_r

and u_z are zero. As a result, the displacement vector components take the form

$$u_r = \frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z}, \quad u_z = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r}, \quad u_\theta = 0, \quad (1)$$

where $\psi \equiv \psi_\theta$. The wave equations describing axisymmetric waves in an elastic medium have the form [3]

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = -k_l^2\phi, \quad (2)$$

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\psi}{r^2} + \frac{\partial^2\psi}{\partial z^2} = -k_t^2\psi. \quad (3)$$

According to the Hooke law, the normal and tangential stresses are expressed through the displacement vector components as

$$\sigma_{rr}^{(0)} = 2\mu\frac{\partial u_r}{\partial r} + \lambda\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right),$$

$$\sigma_{rz}^{(0)} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).$$

We introduce the stress vector $\bar{\sigma}^{(0)} = (\sigma_{rr}^{(0)}, \sigma_{rz}^{(0)})^T$. Then, for the free boundary, the following condition is valid:

$$\bar{\sigma}^{(0)} = 0 \text{ at } r = a. \quad (4)$$

We take into account that the wave of interest, which propagates along the surface, attenuates when it penetrates into the elastic medium. This means that

$$\phi, \psi \longrightarrow 0 \text{ at } r \longrightarrow \infty. \quad (5)$$

Equations (2) and (3) and condition (5) are satisfied by the potentials

$$\phi = A_\phi K_0(k\beta_l r)e^{ikz}, \quad (6)$$

$$\psi = A_\psi K_1(k\beta_t r)e^{ikz}. \quad (7)$$

Here, k is the wave number of the quasi-Rayleigh wave; K_0 and K_1 are zero-order and first-order modified Bessel functions of the second kind, respectively; and A_ϕ and A_ψ are arbitrary constants. In Eqs. (6) and (7), we introduced the dimensionless parameters $\beta_l = \sqrt{1-\xi^2k_l^2/k_t^2}$ and $\beta_t = \sqrt{1-\xi^2}$, where $\xi = k_t/k = c/c_t$ is a dimensionless quantity identical to the ratio of the Rayleigh wave velocity c to the shear wave velocity c_t. We note that an undamped quasi-Rayleigh wave can exist at a free boundary only when β_l and β_t are real, which corresponds to $\xi < 1$. Substituting potentials (6) and (7) in Eqs. (1), we obtain the displacement vector $\bar{u}$ in the form

$$\bar{u} = \begin{pmatrix} u_r \\ u_z \end{pmatrix} = -ke^{ikz}\begin{pmatrix} \beta_l K_1(k\beta_l r) & iK_1(k\beta_t r) \\ -iK_0(k\beta_l r) & \beta_t K_0(k\beta_t r) \end{pmatrix}\bar{A}. \quad (8)$$

Here, we used the notation $\bar{A} = (A_\phi, A_\psi)$. We introduce the elastic impedance matrix of the medium $\mathbf{Z}^{(0)}$ determining the linear relation between the stress $\bar{\sigma}^{(0)}$ and velocity $(-i\omega\bar{u})$ vectors in the absence of the impedance load at the boundary [8]:

$$\bar{\sigma}^{(0)} = -i\omega\mathbf{Z}^{(0)}\bar{u}. \quad (9)$$

As is shown below in Appendix A, the matrix $\mathbf{Z}^{(0)}$ for potentials (6) and (7), which describe an undamped quasi-Rayleigh wave, has the following form at the boundary of the cavity:

$$\mathbf{Z}^{(0)}(\xi, \chi) = i\rho c_t\mathbf{X}^{(0)}(\xi, \chi)$$
$$= i\rho c_t\begin{pmatrix} x_{rr}(\xi, \chi) & -ix_{rz}(\xi, \chi) \\ ix_{rz}(\xi, \chi) & x_{zz}(\xi, \chi) \end{pmatrix}, \quad (10)$$

where the dimensionless elements x_{ij} of the matrix $\mathbf{X}^{(0)}$ have the form

$$x_{rr}(\xi, \chi) = \frac{\xi\beta_t F(\beta_t)F(\beta_l)}{\beta_l\beta_t F(\beta_t) - F(\beta_l)} - \frac{2}{\chi}, \quad (11)$$

$$x_{zz}(\xi, \chi) = \frac{\xi\beta_l}{\beta_l\beta_t F(\beta_t) - F(\beta_l)}, \quad (12)$$

$$x_{rz}(\xi, \chi) = \frac{\xi F(\beta_l)}{\beta_l\beta_t(\beta_t) - F(\beta_l)} + \frac{2}{\xi}. \quad (13)$$

Here, we introduced the function $F(\varepsilon) = K_0(\varepsilon\chi/\xi)/K_1(\varepsilon\chi/\xi)$ and the dimensionless frequency $\chi = k_t a$. Boundary condition (4) means that, at $r = a$, Eq. (9) becomes homogeneous. In this case, it has a nontrivial solution only when the following equation is satisfied:

$$\det\mathbf{Z}^{(0)} = 0 \text{ or } \det\mathbf{X}^{(0)} = 0. \quad (14)$$

Using Eqs. (11)–(13), we represent this equation in the form

$$(2-\xi)^2F(\beta_l) - 4\beta_l\beta_t F(\beta_t) - 2\beta_l\xi^3/\chi = 0. \quad (15)$$

It describes the dispersion properties of a quasi-Rayleigh wave near a cylindrical cavity and coincides with the equation obtained by Biot [3].

Let us proceed to considering the problem stated above, i.e., to the case where an impedance load is present at the boundary. The load causes additional surface stresses described by the vector $\bar{\sigma}^{(L)} = (\sigma_{rr}^{(L)}, \sigma_{rz}^{(L)})^T$. The impedance matrix of such a load

$\mathbf{Z}^{(L)}$ describes the linear relation between the stress $\bar{\sigma}^{(L)}$ and velocity $(-i\omega\bar{u})$ vectors:

$$\bar{\sigma}^{(L)} = -i\omega\mathbf{Z}^{(L)}\bar{u}. \quad (16)$$

For convenience, we introduce the matrix $\mathbf{X}^{(L)}$ and the quantities X_{ij} in the following way:

$$\mathbf{X}^{(L)} = (i\rho c_t)^{-1}\mathbf{Z}^{(L)}, \quad \mathbf{X}^{(L)} = \begin{pmatrix} X_{rr} & -iX_{rz} \\ iX_{rz} & X_{zz} \end{pmatrix}. \quad (17)$$

Because of the linearity of the relations between stresses and velocities in Eqs. (9) and (16), the resulting stress vector $\bar{\sigma}^F$ at the boundary of the cavity is expressed as [5]

$$\bar{\sigma}^{(F)} = \bar{\sigma}^{(0)} + \bar{\sigma}^{(L)}. \quad (18)$$

For the boundary of the cavity, the following boundary condition is valid:

$$\bar{\sigma}^{(F)} = 0 \text{ at } r = a. \quad (19)$$

Substituting stresses (9) and (16) in Eq. (18) and applying boundary condition (19), we obtain a homogeneous equation for determining the vector $\bar{u}$. The condition of existence of a nontrivial solution to this equation leads to a dispersion equation for quasi-Rayleigh waves at the boundary with the impedance load:

$$\det(\mathbf{Z}^{(L)} + \mathbf{Z}^{(0)}) = 0 \text{ or } \det(\mathbf{X}^{(L)} + \mathbf{X}^{(0)}) = 0. \quad (20)$$

In view of Eqs. (10) and (17), we represent it in the form

$$[X_{rr} + x_{rr}(\xi, \chi)][X_{zz} + x_{zz}(\xi, \chi)] = [X_{rz} + x_{rz}(\xi, \chi)]^2. \quad (21)$$

Let us consider the limiting cases. In the absence of the load at the boundary, i.e., $\mathbf{X}^{(L)} = 0$, Eq. (20) takes the form of Eq. (14). In the limit $\chi \longrightarrow \infty$, for the diagonal impedance matrix of the load ($X_{rz} = 0$), Eq. (21) transforms to the dispersion equation for a quasi-Rayleigh wave on a plane in the presence of a two-component impedance load, which was derived in [5].

It is well known that a quasi-Rayleigh wave near a cylindrical cavity has critical frequencies $\omega_{cr} = c_t\chi_{cr}/a$ such that a wave with a frequency lower than ω_{cr} cannot propagate in it. Let us consider the dependence of χ_{cr} on the impedance load. The frequency is critical if the velocity of the quasi-Rayleigh wave for it is identical to the velocity of shear waves in the elastic medium, i.e., at $\xi = 1$. Substituting this value in Eq. (21), we obtain

$$(X_{rr} - 2\chi_{cr}^{-1})(X_{zz} - \beta_l^{cr}K_1(\chi_{cr}\beta_l^{cr})K_0^{-1}(\chi_{cr}\beta_l^{cr})) = (X_{rz} + 1)^2, \quad (22)$$

where $\beta_l^{cr} = \sqrt{1 - k_l^2/k_S^2}$ is a constant depending on the elastic properties of the medium. We note that loads can exist with $\chi \longrightarrow \infty$, for which the propagation of quasi-Rayleigh waves is impossible for any frequencies, except for the case of waves on a plane. This follows from the analysis of Eq. (22) in the limit $\chi_{cr} \longrightarrow \infty$:

$$X_{rr}(X_{zz} - \beta_l^{cr}) = (X_{rz} + 1)^2, \quad (23)$$

which gives the equation for determining this kind of load.

Now let us determine the type of load that provides preset properties of quasi-Rayleigh waves. For illustration and for convenience, we consider the diagonal impedance matrices ($X_{rz} = 0$). In this case, Eq. (21) can be represented in the form

$$[X_{rr} + x_{rr}(\xi, \chi)][X_{zz} + x_{zz}(\xi, \chi)] = \zeta(\xi, \chi), \quad (24)$$

where $\zeta(\xi, \chi) = [x_{rz}(\xi, \chi)]^2$. Evidently, at fixed values of ξ and χ, it describes a hyperbola in the (X_{rr}, X_{zz}) plane. Thus, for any point (ξ', χ') (or any region in the (ξ, χ) plane), there always is a hyperbola (or region, respectively) in the impedance plane (X_{rr}, X_{zz}) such that, in the presence of a load belonging to this hyperbola (or region), the dispersion curve of the quasi-Rayleigh wave passes through this point (or region in the (ξ, χ) plane). In [5], similar hyperbolas were considered for the case of quasi-Rayleigh waves on a plane boundary. For example, we assume that the point (ξ', χ') belongs to the dispersion curve of the quasi-Rayleigh wave in the absence of the surface load; i.e., it is the solution to Eq. (15). In this case, Eq. (24) will describe a hyperbola lying in the (X_{rr}, X_{zz}) plane and passing through the point (0, 0) corresponding to zero load; in this case, Eq. (24) takes the form

$$(X_{rr}/x_{rr}(\xi, \chi) + 1)(X_{zz}/x_{zz}(\xi, \chi) + 1) = 1.$$

Let us consider the behavior of the hyperbolas described by Eq. (24) and their dependence on the choice of the point (ξ', χ'). First, we consider the effect of the quantity $\xi' \in (0, 1]$ at a fixed value of χ'. This effect is illustrated in Fig. 1 for the case of $\chi' = 2.5$. Hyperbolas *1*, *2*, and *3* describe the impedances X_{rr} and X_{zz} at which the characteristics of quasi-Rayleigh waves pass through the points (1, χ'), (χ_c, χ') (where $\xi_c = 0.58$), and (0.35, χ'), respectively. One can see that, as the parameter ξ' decreases from 1 to 0, the corresponding hyperbolas at a fixed χ' shift rightward and upward, beginning from curve *1*, to the hyperbolas with X_{rr}, $X_{zz} \longrightarrow \infty$. It should be noted that, in the impedance plane (X_{rr}, X_{zz}), three regions are formed (regions I, II, and III in Fig. 1). For the load belonging to region I, which lies below the left-hand branch of hyperbola *1*, the propagation of a quasi-Rayleigh wave with a dimensionless frequency lower than χ' is impossible. In the second region lying between the branches of hyperbola *1*, only one quasi-Rayleigh wave propa-

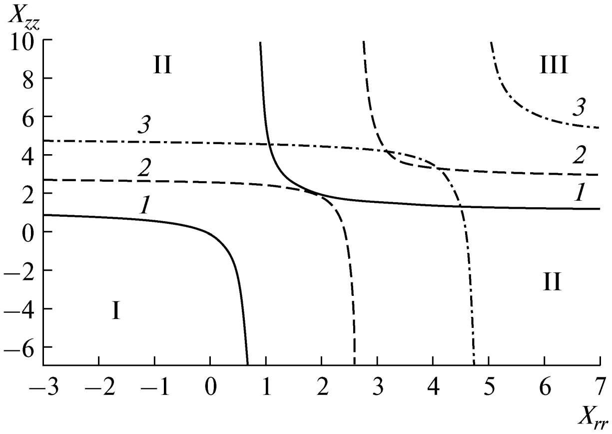

Fig. 1. Hyperbolas in the (X_{rr}, X_{zz}) plane at a fixed value of $\chi' = 2.5$ and at $\sigma = 0.4$ for different values of ξ: ξ = (*1*) 1, (*2*) ξ_c, and (*3*) 0.35.

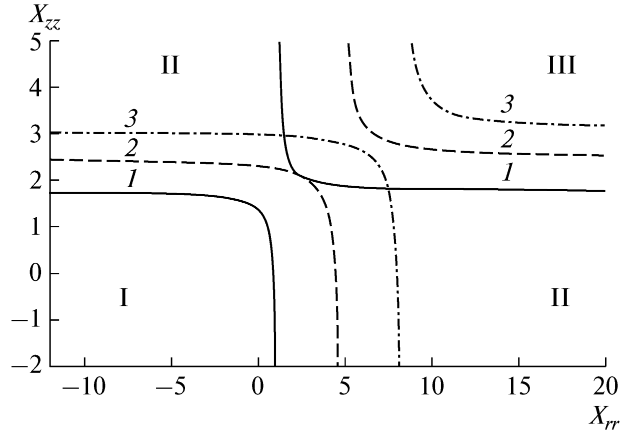

Fig. 2. Hyperbolas in the (X_{rr}, X_{zz}) plane at a fixed value of $\xi' = 0.8$ and at $\sigma = 0.4$ for different values of χ: χ = (*1*) ∞, (*2*) χ_c, and (*3*) 0.25.

gates, its dispersion curve possessing a certain velocity $\xi' \in (\xi_c, 1]$ at χ'. Finally, when the load belongs to region III, two surface waves propagate with different velocities ξ^{I} and ξ^{II} at χ'. Hyperbola *2* corresponds to such a value of $\xi' = \xi_c$ at which its left-hand branch touches the right-hand branch of hyperbola *1*. This means that, if Eq. (24) describes the behavior of two quasi-Rayleigh waves and the dimensionless velocity of one of them reaches the value $\xi^{I} = 1$ at χ', then, at any load lying in region III, the other wave at a dimensionless frequency χ' will propagate with a velocity $\xi^{II} < \xi_c$.

Now, in Eq. (24), we take a fixed value of ξ' and consider the effect of the parameter $\chi' \in (0, \infty)$ on the behavior of the corresponding hyperbolas in the (X_{rr}, X_{zz}) plane. This behavior is illustrated in Fig. 2 for $\xi' = 0.8$. Hyperbolas *1*, *2*, and *3* describe the impedances X_{rr} and X_{zz}, at which the characteristics of the quasi-Rayleigh wave pass through the points (ξ', ∞), (ξ', χ_c), where $\chi_c = 0.46$, and $(\xi', 0.25)$, respectively. As χ' decreases from ∞ to 0, the hyperbolas shift upward and rightward from hyperbola *1* to the values X_{rr}, $X_{zz} \longrightarrow \infty$, which correspond to the point $(\xi', 0)$. As in the first case, we consider three regions (I, II, and III) in Fig. 2. The branches of hyperbola *1* bound region II, for which only one quasi-Rayleigh wave is possible with the velocity ξ'. In the presence of a load belonging to region I (below the left-hand branch of hyperbola *1*), the quasi-Rayleigh wave with the velocity ξ' does not exist, whereas, in region III, two such waves propagate with different values of χ^{I} and χ^{II} at ξ'. Hyperbola *2* touches hyperbola *1* at a certain point. The value of χ_c corresponds to the maximal value of χ' of one of the waves, for which the propagation of two quasi-Rayleigh waves is possible with the velocity ξ'. We note that, as an example, here and below, the Poisson's ratio of the elastic medium, σ, was chosen to be 0.4, which corresponds to such materials as lead or glass. The method of determining the values of χ_c and ξ_c is described below in Appendix B.

It is of interest to consider the specific case $(\xi, \chi) = (1, \infty)$, which corresponds to a quasi-Rayleigh wave propagating over a plane with the velocity of shear waves. The dispersion curve passing through this point is described by Eq. (23). If $X_{rz} = 0$, the corresponding equation will describe a hyperbola in the (X_{rr}, X_{zz}) plane, which divides this plane into three regions. If we use the load parameters below the left-hand branch of the hyperbola, no quasi-Rayleigh waves can propagate, no matter what their frequencies are. For impedances lying in the region between the branches of the hyperbola, propagation of one quasi-Rayleigh wave is possible at certain frequencies. If the load belongs to the region above the right-hand branch of the hyperbola, at certain frequencies we obtain a simultaneous propagation of two quasi-Rayleigh waves. One of them can have a dispersion characteristic passing through any point of the (ξ, χ) plane, whereas the dispersion characteristic of the other wave should always lie below the curve described by the equation (see Appendix B)

$$(1 + \sqrt{\zeta(\xi, \chi)})^2 = x_{rr}(\xi, \chi)(\beta_l^{cr} + x_{zz}(\xi, \chi)).$$

As examples, we consider several impedance load models that illustrate their influence on the properties of quasi-Rayleigh waves. The firs model is a cylindrical cavity filled with a nonviscous fluid. The density of the fluid is ρ_f, and the wave number of elastic waves in it is k_f. We represent the fluid as an impedance load. For it, in the matrix $\mathbf{X}^{(L)}$, the only nonzero element is X_{rr}. The acoustic impedance Z_f of the fluid is identical to the ratio of the normal pressure in the fluid to the radial velocity [9]. Taking into account the definition of the matrix $\mathbf{X}^{(L)}$, we determine the element X_{rr}:

$$X_{rr} = \xi \frac{\rho_f I_0(k\alpha a)}{\rho \alpha I_1(k\alpha a)}, \tag{25}$$

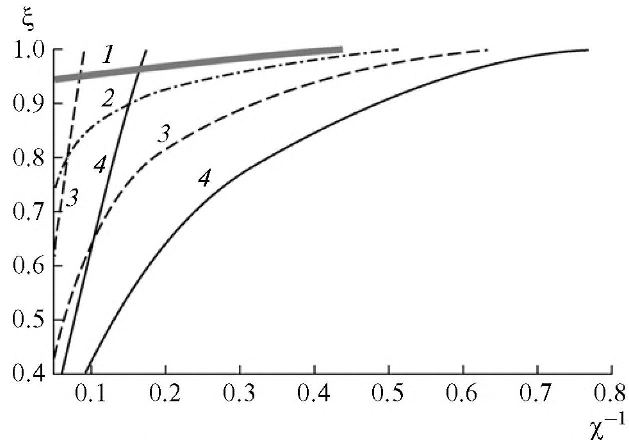

Fig. 3. Dispersion characteristics of quasi-Rayleigh waves in the presence of impedance loads given by Eq. (32) at $\sigma = 0.4$ for different values of l/a and g: $l/a =$ (*1*) 0, (*2*, *3*) 0.05, and (*4*) 0.1; $g =$ (*2*) 1 and (*3*, *4*) 3.

where $\alpha = \sqrt{1 - \xi^2 k_f^2/k_t^2}$. Here, I_0 and I_1 are the zero- and first-order modified Bessel functions of the first kind, respectively. Substituting of Eq. (25) in Eq. (20) yields the dispersion equation

$$(2-\xi)^2 F(\beta_l) - 4\beta_l\beta_t F(\beta_t)$$
$$-\beta_l \xi^3 \left[\frac{2}{\chi} - \xi\frac{\rho_f I_0(\alpha\chi/\xi)}{\alpha\rho I_1(\alpha\chi/\xi)}\right] = 0.$$

In the fluid-filled cavity, when $k > k_f$, this equation describes the Stoneley waves monotonically attenuating along the radius in the direction from the boundary into the fluid or into the solid; when $k < k_f$, it describes the modes "oscillating" along the radius in the fluid and monotonically attenuating in the solid. The dispersion equations for these waves have been derived and analyzed in many publications, e.g., in [3]. The above consideration describes the alternative method of their derivation, as compared to the method used in the aforementioned publications.

Let us consider another load model. We assume that the surface of the solid is covered with a thin cracked layer with the density $\rho^{(L)}$ and height l, the latter being small compared to the shear wavelength in the elastic medium: $l \ll \lambda_t$. The characteristic transverse size d of the cuts of the medium in this layer is assumed to be small: $d \ll \lambda_t$. In addition, we assume that the shear waves' velocities in the layer and in the medium are identical. The load impedance matrix has the form [5, 6]

$$\mathbf{X}^{(L)} = g\begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix}, \qquad (26)$$

where $\Omega = k_t l$ and $g = \rho^{(L)}/\rho$. To estimate the limits of applicability of this model, we assume that the height of the layer is much smaller than both the radius of the cavity and the wavelength: $l \ll a$ and $l \ll \lambda_t$. Then, multiplying both inequalities by k_t, we obtain the applicability conditions in the form $\Omega \ll \chi$ and $\Omega \ll 2\pi$.

Figure 3 shows the dependences of the dimensionless velocity ξ on the dimensionless quantity χ^{-1} for quasi-Rayleigh waves in the case of the load described by Eq. (26) for different values of l/a (or the identical quantity Ω/χ) and g. In the case of $\Omega = 0$, which corresponds to the absence of load (curve *1*), the dispersion characteristic of the quasi-Rayleigh wave is described by Eq. (15). Curves *2* and *3* correspond to $\Omega/\chi = l/a = 0.05$ and $g = 1$ and 3, respectively. Curve *4* corresponds to $l/a = 0.1$ and $g = 3$. One can see that, when the parameter l/a or g increases, the effect of the load on the properties of quasi-Rayleigh waves grows. For example, on the surface of the cavity, two quasi-Rayleigh waves can exist and, the greater the ratio l/a, the smaller the critical frequency of each of these waves. According to the conditions of applicability formulated above for the model under consideration, at $\Omega \neq 0$, the dispersion curves most adequately predict the behavior of quasi-Rayleigh waves in the low-frequency range, i.e., at large values of χ^{-1}.

Thus, in this paper, we considered the properties of quasi-Rayleigh waves near a cylindrical cavity in the presence of a surface impedance load. By means of the impedance method, we derived the dispersion equation describing the behavior of such waves and showed that this equation can be represented by the condition that the determinant of the sum of the impedance matrices of the load and the medium is zero. In various limiting cases, this equation takes the forms of well-known results [3, 5]. Further analysis of this equation allowed us to determine the dependence of the critical frequency of the quasi-Rayleigh wave on the cavity boundary load. In particular, we showed that there exists a load such that it permits the presence of a quasi-Rayleigh wave only in the case of a plane boundary. We also considered the problem of choosing the two-component impedance load so that the dispersion curve of the quasi-Rayleigh wave passed through a given point (ξ', χ'). We determined the corresponding condition for this load. It describes hyperbolas in the (X_{rr}, X_{zz}) plane. The influence of the chosen point (ξ', χ') on the position of such a hyperbola was analyzed in detail. It was shown that the impedance plane is divided into three regions, and, depending on the region to which the load belongs, we obtained either the absence of quasi-Rayleigh waves near the cavity or the presence of one wave or two waves. In the latter case, the analysis of the hyperbolas showed that the choice of the point through which the dispersion curve of one of the waves passes imposes limitations on the dispersion properties of the other wave. To illustrate the aforementioned results, we considered several models of the impedance load. For example, for the load represented by a fluid filling the cylindrical cavity, we obtained the well-known dispersion equation for Stoneley waves in such a cavity [3]. We also studied the behavior of quasi-Rayleigh waves for the load model in the form of a cut layer on the cavity surface. The results of our studies can be used in the design of devices for controlling the properties of quasi-Rayleigh waves in systems with cylindrical geometry (pipelines, boreholes, borehole instruments, etc.).

APPENDIX A

THE IMPEDANCE MATRIX OF THE MEDIUM

Here, we determine the form of the impedance matrix of the medium for potentials (6) and (7). By definition, this matrix $\mathbf{Z}^{(0)}$ is determined by formula (9) and determines the linear relation between the stress and displacement vectors [8]. According to Eq. (8), the displacement vector $\bar{u}$ linearly depends on the vector $\bar{A}$:

$$\bar{u} = -ke^{ikz}\begin{pmatrix} \beta_l K_1(k\beta_l r) & iK_1(k\beta_t r) \\ -iK_0(k\beta_l r) & \beta_t K_0(k\beta_t r) \end{pmatrix}\bar{A} = \mathbf{T}\bar{A}. \quad (A.1)$$

The dependence of the stress vector $\bar{\sigma}^{(0)}$ on $\bar{A}$ is linear and well known (see, e.g., [3]):

$$\bar{\sigma} = \mathbf{N}\bar{A} = 2\rho c_t^2 k^2 e^{ikz}\begin{pmatrix} K_0(k\beta_l r)[1-\xi^2/2] + \dfrac{\beta_l K_1(k\beta_l r)}{kr} & i\beta_t K_0(k\beta_t r) + i\dfrac{K_t(k\beta_t r)}{kr} \\ -i\beta_l K_1(k\beta_l r) & K_1(k\beta_t r)[1-\xi^2/2] \end{pmatrix}\bar{A}. \quad (A.2)$$

Expressing $\bar{A}$ through $\bar{u}$ with the use of Eq. (A.1) and substituting the result in Eq. (A.2), we obtain the linear relation between the stress vector and the vector $\bar{u}$: $\bar{\sigma} = \mathbf{NT}^{-1}\bar{u}$. The matrix $\mathbf{T}^{-1}$ has the form

$$\mathbf{T}^{-1} = -\frac{e^{-ikz}}{kK_1(k\beta_l r)K_1(k\beta_t r)[\beta_l\beta_t F(\beta_t) - F(\beta_l)]}\begin{pmatrix} \beta_t K_0(k\beta_t r) & -iK_1(k\beta_t r) \\ iK_0(k\beta_l r) & \beta_l K_1(k\beta_l r) \end{pmatrix}. \quad (A.3)$$

Now, using Eq. (9), we obtain the impedance matrix of the medium $\mathbf{Z}^{(0)}$ in the form

$$\mathbf{Z}^{(0)} = \mathbf{NT}^{-1} = i\rho c_t\begin{pmatrix} x_{rr}(\xi,\chi) & -ix_{rz}(\xi,\chi) \\ ix_{rz}(\xi,\chi) & x_{zz}(\xi,\chi) \end{pmatrix}, \quad (A.4)$$

where the functions $x_{ij}(\xi, \chi)$ are determined by Eqs. (11)–(13). Note that the impedance matrix of the medium proved to be Hermitian.

APPENDIX B

THE CONDITIONS OF SIMULTANEOUS PROPAGATION OF TWO QUASI-RAYLEIGH WAVES

As it was shown above, there exists a region of impedances X_{rr} and X_{zz} (e.g., regions III in Figs. 1 and 2) such that the choice of the impedance within it leads to the simultaneous propagation of two quasi-Rayleigh waves. From Eq. (24), it is evident that the characteristics of these waves do not intersect. Thus, the dimensionless velocity ξ of one of the waves (this wave is indicated by the index (F) in subsequent calculations) with at any frequencies exceed the velocity of the other wave (indicated by the index (S)). Below, we will show that, knowing the properties of the faster wave, we can determine certain limitations for the properties of the other wave. We assume that the load is given by the impedance matrix of type (17) with $X_{rz} = 0$. In this case, the dispersion equation of the quasi-Rayleigh wave is given by Eq. (24), which describes the hyperbolas in the (X_{rr}, X_{zz}) plane. If two quasi-Rayleigh waves are present, their hyperbolas may intersect or touch each other. Let us consider the case in which two hyperbolas have one common point (X_{rr}^c, X_{zz}^c), i.e., are tangent to one another. In Eq. (24), for each of the hyperbolas for the (F) and (S) waves, we express X_{zz}^c as follows:

$$X_{zz}^c = \zeta^{(F),(S)}(X_{rr}^c + x_{rr}^{(F),(S)})^{-1} - x_{zz}^{(F),(S)}. \quad (B.1)$$

The derivative dX_{tt}/dX_{nn} at the tangency point for each of the hyperbolas is given by the expression

$$(dX_{zz}/dX_{rr})_c = -\zeta^{(F),(S)}(X_{rr}^c + x_{rr}^{(F),(S)})^{-2}. \quad (B.2)$$

Using Eqs. (B.1) and (B.2), we express the condition that the hyperbolas are tangential to each other at the point (X_{rr}^c, X_{zz}^c) in the form

$$(\sqrt{\zeta^{(F)}} + \sqrt{\zeta^{(S)}})^2 = (x_{rr}^{(F)} - x_{rr}^{(S)})(x_{zz}^{(F)} - x_{zz}^{(S)}). \quad (B.3)$$

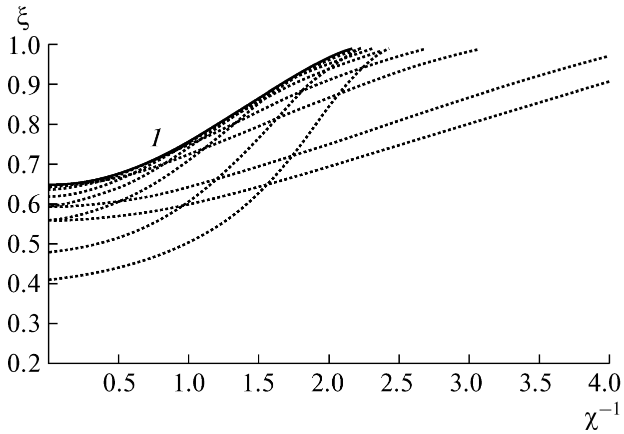

Fig. 4. Dispersion characteristic described by Eq. (B.5) (curve *1*) and the dispersion curves of (*S*) waves (the dotted lines) for different load impedances allowing the existence of two quasi-Rayleigh waves.

This means that, if we know the dispersion characteristic of one wave, e.g., the (F) wave, we can determine the limitation on the dispersion properties of the other wave. Such a condition arises as a result of substituting the corresponding coefficients $x_{ii}^{(F)}$ and $\zeta^{(F)}$ in Eq. (B.3). Let us consider the two cases described above. In the first of them, the dispersion curve of the (F) wave reaches the value $\xi^{(F)} = 1$ at $\chi^{(F)} = \chi'$. Substituting these values in Eq. (24), we obtain a hyperbola in the (X_{rr}, X_{zz}) plane, and, substituting the values $\xi^{(S)} = \xi_c$ and $\chi^{(S)} = \chi'$, we obtain the hyperbola corresponding to the (S) wave. The value of ξ_c is maximal, if the hyperbolas of the two waves are tangential to each other. Substituting $\xi^{(F)}$, $\xi^{(S)}$, and χ' in Eq. (B.3), we obtain the equation that implicitly depends on ξ_c at $\chi^{(S)} = \chi'$:

$$(1+\sqrt{\zeta(\xi_c,\chi')})^2 = (2\chi_{\rm cr}^{-1} + x_{rr}(\xi_c,\chi'))$$
$$\times(\beta_l^{\rm cr} K_1(\chi'\beta_l^{\rm cr})K_0(\chi'\beta_l^{\rm cr})^{-1} + x_{zz}(\xi_c,\chi')).$$

The value of ξ_c obtained from this equation is the maximal possible velocity of the (S) wave at the frequency $\omega' = c_t\chi'/a$ and at any impedance load for which ω' is the critical frequency of the (F) wave.

In the second case, we assume that $\xi^{(F)} = \xi'$ at $\chi^{(F)} \longrightarrow \infty$. For the (S) wave, the maximal value of $\chi^{(S)}$ at ξ' is χ_c. The hyperbola obtained for the (F) wave from Eq. (24) for the point $\xi^{(F)} = \xi'$, $\chi^{(F)} \longrightarrow \infty$ touches with its right-hand branch the hyperbola that corresponds to the point $\xi^{(S)} = \xi'$, $\chi^{(S)} = \chi_c$. For the values of $\chi^{(S)}$ and $\chi^{(F)}$ at ξ', Eq. (B.3) has the form

$$\left(\left|\frac{(2-\xi')-2\beta_l'\beta_t'}{\xi'(1-\beta_l'\beta_t')}\right| + \sqrt{\zeta(\xi',\chi_c)}\right)^2$$
$$= \left(\frac{\xi'}{\beta_l'^{-1}-\beta_t'} + x_{zz}(\xi',\chi_c)\right)\left(\frac{\xi'}{\beta_t'^{-1}-\beta_l'} + x_{rr}(\xi',\chi_c)\right), \quad \text{(B.4)}$$

where β_l' and β_t' are the values of the corresponding parameters for ξ'. From this equation, by numerical methods, we can determine χ_c and ω_c. The latter is the maximal frequency value at which the slower wave can propagate with the velocity ξ' under the condition that the (F) wave propagates with this velocity at $\chi^{(F)} \longrightarrow \infty$.

Now, let us determine the properties of the (S) wave if we know that the dispersion curve of the (F) wave reaches the value $\xi^{(F)} = 1$ at $\chi^{(F)} \longrightarrow \infty$ (such an (F) wave will propagate only on the plane with the shear wave velocity). As was shown above, the hyperbola lying in the (X_{rr}, X_{zz}) plane and corresponding to $\xi^{(F)}$ and $\chi^{(F)}$ is the limiting one for all the possible hyperbolas. For the values $\xi^{(F)}$ and $\chi^{(F)}$, Eq. (B.4) takes the form

$$(1+\sqrt{\zeta^{(S)}})^2 = x_{zz}^{(S)}(\beta_l^{\rm cr} + x_{rr}^{(S)}). \quad \text{(B.5)}$$

In the (ξ, χ) plane, it determines the dispersion curve (curve *1* in Fig. 5). If the diagonal load impedance matrix is such that two quasi-Rayleigh waves arise, the dispersion characteristic of one of them will always lie below this curve, irrespective of the specific values of the load impedance matrix elements. The curves represented by the dotted lines in Fig. 5 show different dispersion characteristics for such slower quasi-Rayleigh waves corresponding to different values of the load impedance matrix elements. One can see that curve *1* is the envelope of the entire series of these dispersion characteristics. In a similar way, it is always possible to determine the corresponding envelope for the (S) waves in the case in which the dispersion curve of the (F) wave passes through a certain point (ξ', χ').

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